Sorting: an update

Daniel J. Bernstein

Sorting arrays of secret integers

Cryptosystems that use a sorting subroutine:

- Classic McEliece: various deployments
- NTRU-HPS: another candidate for ISO
- NTRU Prime: now in 13 SSH implementations
- LESS: candidate for NIST
- PERK: candidate for NIST
- CTIDH: non-interactive key exchange
- More that I've noticed: Round2, LEDA, BIG QUAKE, GeMSS, PKP-DSS, WAVE

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Also many higher-level security protocols: shuffling votes, shuffling network packets, etc.

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No, radix sort has data-dependent array indices.

Can attackers deduce the secret data from branch timings or cache timings, as in many other timing attacks? Maybe! Unnecessary attack surface.

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Can these attacks be pushed further? Maybe!

Constant-time sorting

Bubblesort to the rescue:

```
void int32_sort(int32_t *x,long long n)
{ for (long long j = n;j > 1;--j)
    for (long long i = 1;i < j;++i)
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Bubblesort is an example of a **sorting network**: a sequence of `minmax` at predictable positions.

Speed

A faster sorting network

```
void int32_sort(int32_t *x,long long n)
{ if (n < 2) return;
  long long t = 1; while (t < n - t) t += t;
  for (long long p = t;p > 0;p >>= 1) {
    for (long long i = 0;i < n - p;++i)
      if (!(i & p))
        crypto_int32_minmax(x+i,x+i+p);
    for (long long q = t;q > p;q >>= 1)
      for (long long i = 0;i < n - q;++i)
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What is this algorithm?

Previous slide: C translation of 1973 Knuth "merge exchange", which is a simplified version of 1968 Batcher "odd-even merge" sorting network.

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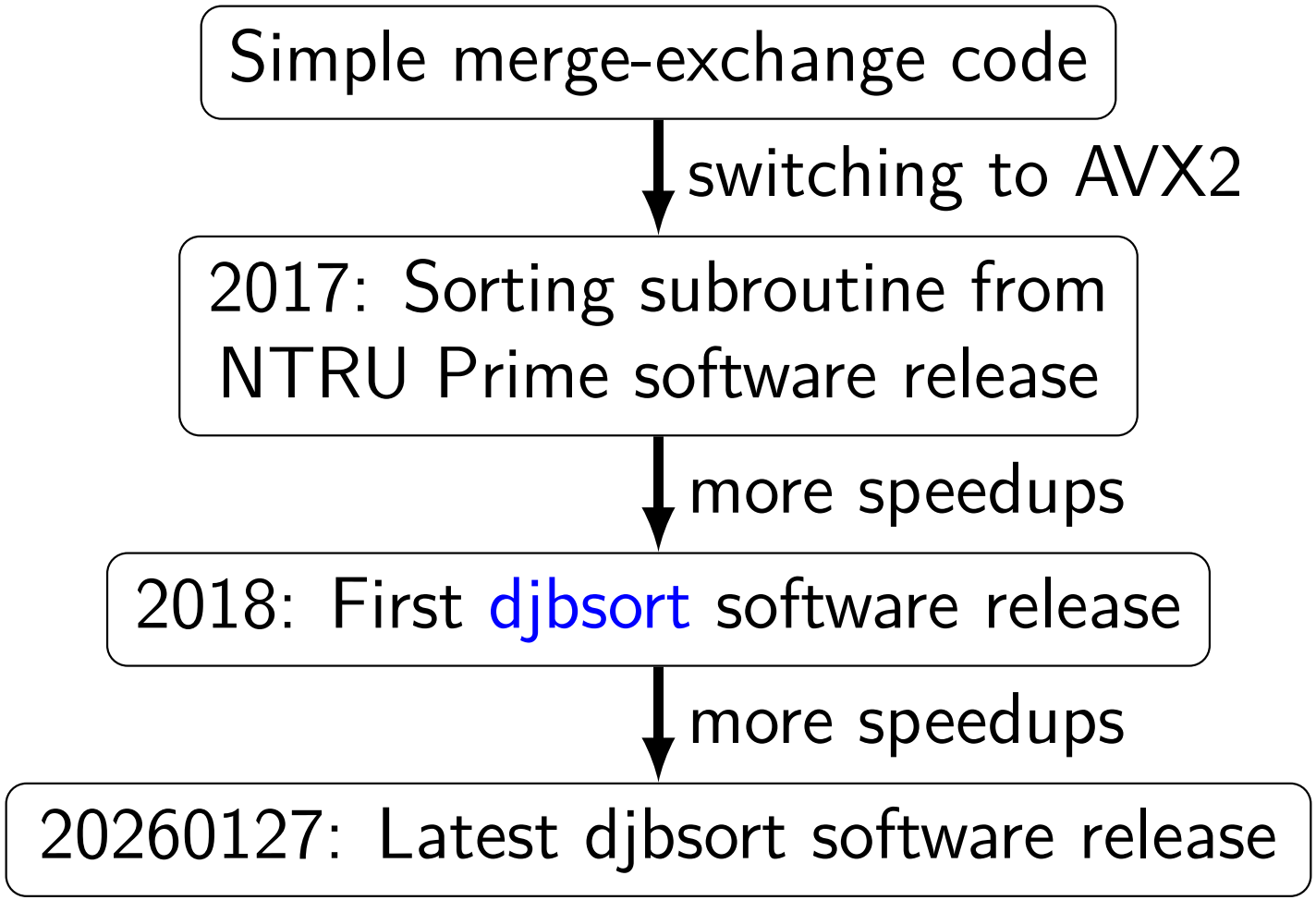
Many sorting-network descriptions are more limited: e.g., requiring n to be a power of 2.
Also, Wikipedia says "Sorting networks ...
are not capable of handling arbitrarily large inputs".

Even faster constant-time sorting



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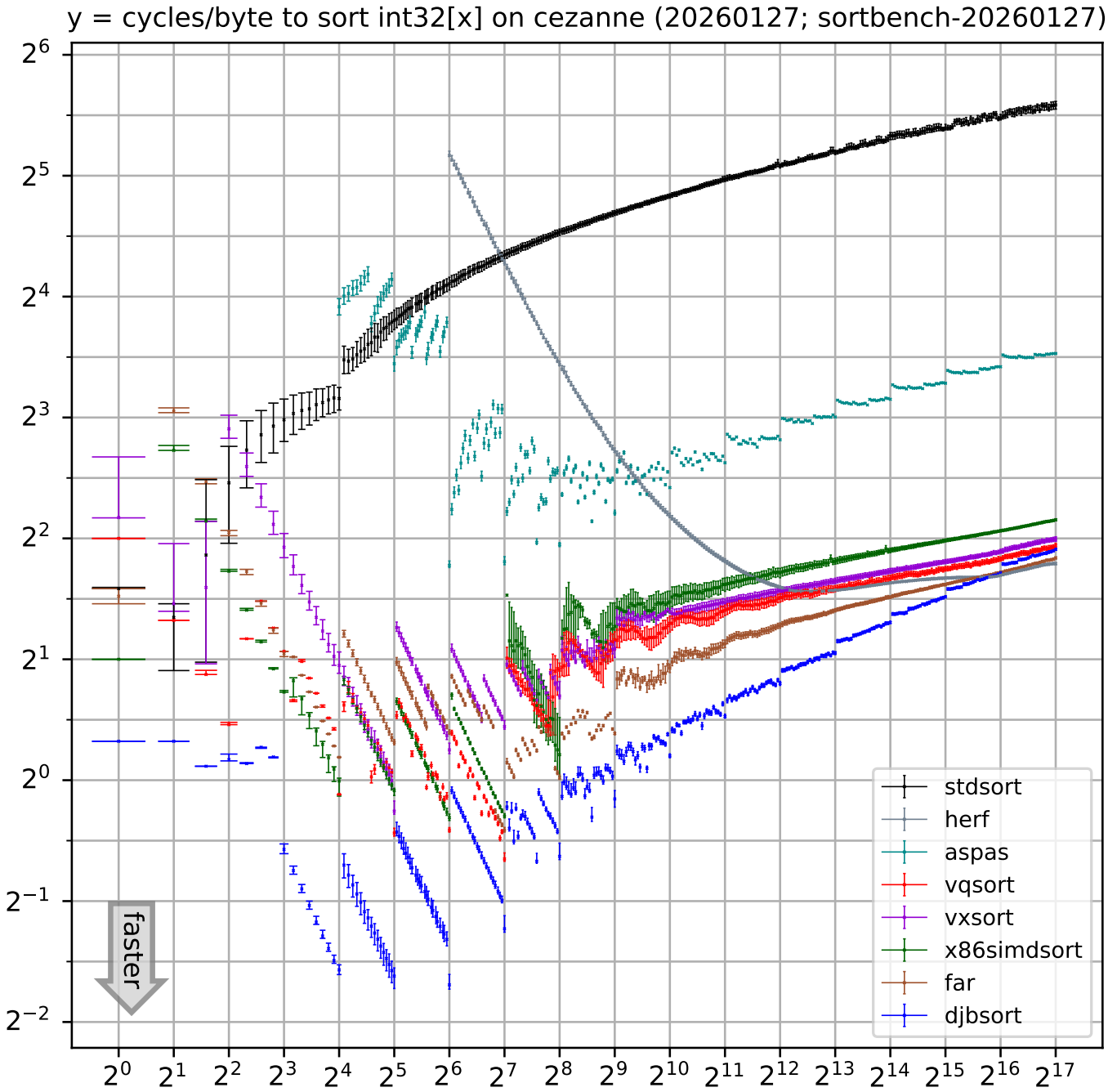
Comparison platform: one core of an AMD Ryzen 5 PRO 5650G (2021 CPU launch; Zen 3 microarch).


y = cycles/byte to sort int32[x] on cezanne (20260127; sortbench-20260127)
2^6
2^5
2^4
2^3
2^2
2^1
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2^{-1}
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faster
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far, vxsort, vqsort, x86simdsort use sorting networks for small n; semi-vectorized quicksort for larger n.
`std::sort` uses insertion sort for small n;
quicksort for large n. (Heapsort if quicksort stalls.)

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Security

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If a memory-safety test passes for one size-n array, and the code doesn't inspect array alignment etc., then the code is memory-safe for all size-n arrays. (Unlike vxsort, which crashes on *some* inputs. Does vxsort allow data leaks? Stack smashing? Maybe.)

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— I have a `clang` patch that removes many of the branch-introducing screwups in that compiler. Patch already adopted by Fil-C. But universal rollout will take time; "many" is unlikely to be all; and `clang` will probably keep adding new screwups.

Another compiler modification

A 2018 paper complained that OpenSSL had "37 different functions" for constant-time computations. Patched `clang` for `__builtin_ct_choose`.

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So `__builtin_ct_select(a<b,a,b)` is safe?

No: `a<b` can already generate a branch!
This is why NaCl's coding rules say "always assume that a comparison in C is compiled into a branch".
What people actually need is safe comparison operations, as in the `cryptoint` API.

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This achieves path coverage for djbsort's code once the test runs cover enough array sizes. The test suites in libmceliece etc. cover the specific array sizes needed for those cryptosystems.

Take the compiler out of the loop?

The djbsort code is generated by a Python script. Could modify this to generate assembly directly.

Maybe preferable: generate code in the Jasmin language, which has various security features including `secret` data types.

Verification

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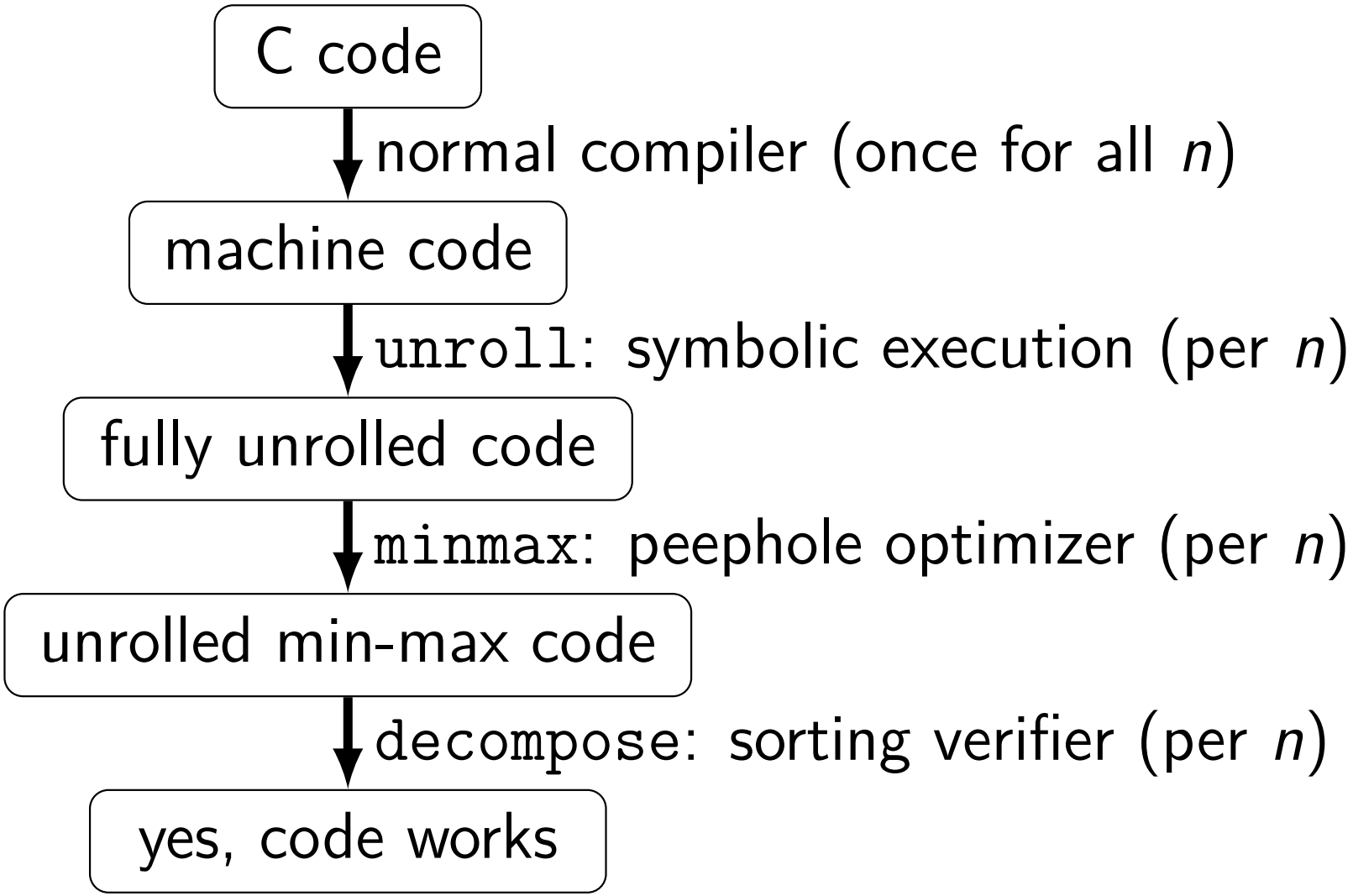
Random-input correctness tests under `valgrind` make clear that it's *close* to a sorting network:

- The code is memory-safe for all size-n inputs.
- The output is correct for *most* size-n inputs.

But could occasional inputs be mis-sorted?
Maybe this is a problem for the application!

Verification pipeline for djbsort



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Do these tools have bugs? Does angr have bugs? Switching to HOL Light would reduce risks.

Testing vs. verification

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void int32_sort(int32_t *x,long long n)
{ for (long long j = n;j > 1;--j)
    for (long long i = 1;i < j;++i)
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Can write tests to catch this; verification is easier!