Hash-based signatures I

Hash functions and one-time signatures

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1 April 2022

Python snippets for this talk:
https://cr.yp.to/talks/2022.04.01-2/ots-20220401.tar.gz

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```
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The `sha256sum` program computes the SHA-256 hash function.
This is a function $H : \{0,1\}^* \to \{0,1\}^{256}$. Each output is 32 bytes.

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This is a function $H : \{0,1\}^* \to \{0,1\}^{256}$. Each output is 32 bytes.

(Actually, SHA-256 requires input to be at most $2^{64} - 1$ bits.
Exercise: Compute $\#$ years for today's fastest CPU to reach this limit.)

The SHA-256 cryptographic hash function in Python 3

```
>>> import hashlib
>>> def sha256(x):
...   h = hashlib.sha256()
...   h.update(x)
...   return h.digest()
...
>>> print(sha256(b'hello').hex())
2cf24dba5fb0a30e26e83b2ac5b9e29e1b161e5c1fa7425e73043362938b9824
>>>
```

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5891b5b522d5df086d0ff0b110fbd9d21bb4fc7163af34d08286a2e846f6be03
>>> print(sha256(b'hello\n'*1000000).hex())
1a2cce61984891495b00826ef591104a34ff35766bbbcaaff965f766154812ab
>>>
```

Goals of cryptographic hash functions

What do we want from a hash function $H : \{0,1\}^* \rightarrow \{0,1\}^n$?

For any string x, think of $H(x)$ as an n-bit fingerprint of x.

Goals:

- $H(x)$ looks totally random;
- nobody can find two different strings x, x' with $H(x) = H(x')$;
- any tiny change from x to x' makes a totally new $H(x')$;
- nobody can compute $H(x)$ without knowing all of x;
- nobody can compute a secret x given only $H(x)$;
- ...

Warning: Some hash goals are difficult to mathematically define.

Generic hardness of preimage resistance

Goal: Given $y \in H(\{0,1\}^*)$, finding $x \in \{0,1\}^*$ with $H(x) = y$ is hard.

Here y is given, and is known to be the image of some $x \in \{0,1\}^*$.
Typically there are many such x,
but it should be computationally hard to find any.

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Generic attack: Try $\approx 2^n$ random choices of x.
If the output of H is distributed uniformly then
each x has a $1/2^n$ chance of $H(x) = y$.

e.g. $\approx 2^{128}$ tries if $n = 128$: very expensive.

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Exercise: Given $y_1, y_2, \ldots, y_{2^{20}}$,
how long does it take to find $x_1, x_2, \ldots, x_{2^{20}}$
such that $H(x_1) = y_1$ and $H(x_2) = y_2$ and $\ldots$ and $H(x_{2^{20}}) = y_{2^{20}}$?

Generic hardness of second-preimage resistance

Goal: Given $x \in \{0,1\}^*$, finding $x' \in \{0,1\}^*$ with $x \neq x'$ and $H(x') = H(x)$ is hard.

Here x is given, determining $y = H(x)$.
Typically there are many other $x' \neq x$ with the same image, but it should be computationally hard to find any.

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Typically there are many other $x' \neq x$ with the same image, but it should be computationally hard to find any.

Generic attack: Try $\approx 2^n$ random choices of $x' \neq x$.
Same speed as for first preimages.

Generic hardness of collision resistance

Goal: Finding $x, x' \in \{0,1\}^*$ with $x \neq x'$ and $H(x') = H(x)$ is hard.

Attacker has full flexibility to choose any output y.
It should still be computationally hard
to find two different strings x, x' with the same output.

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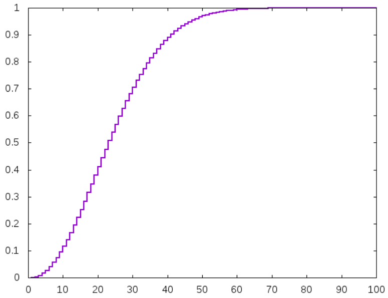
Generic attack: Try $\approx 2^{n/2}$ random choices of x. This number is much lower than the other two because there is no restriction on the target.

The "birthday paradox": if one draws $\approx 1.17\sqrt{m}$ elements at random from a set of m elements, then with 50% probability one has picked one element twice.



Weaknesses in common cryptographic hash functions

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SHA-1 (1995 NSA): $n = 160$, so 2^{80} generic collision attack.
Collisions published (2017): https://shattered.io/.
Practical attack, chosen-prefix collision (2020):
https://sha-mbles.github.io/

The NSA view of cryptographic standardization

"Narrowing the encryption problem to a single, influential algorithm might drive out competitors, and that would reduce the field that NSA had to be concerned about. Could a public encryption standard be made secure enough to protect against everything but a massive brute force attack, but **weak enough to still permit an attack of some nature** using very sophisticated (and expensive) techniques?"

(Emphasis added.)

This quote is from an internal NSA history book.

Some unbroken hash functions

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SHA-512 (2001 NSA): $n = 512$.

"SHA-2" refers to SHA-256, SHA-512, etc.

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Another popular SHA-3 finalist, faster than SHA-3 in software: BLAKE.

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Everyone learns signer's public key.

Using secret key, signer can sign any message m,
producing a signed message (m, s).

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Attacker looks at public key and at signed messages.
Tries modifying the signed messages or creating new messages.

A signature scheme for empty messages: key generation

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```
import os,hashlib

def sha3_256(x):
  h = hashlib.sha3_256()
  h.update(x)
  return h.digest()

def keypair():
  secret = sha3_256(os.urandom(32))
  public = sha3_256(secret)
  return public,secret
```

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  secret = sha3_256(os.urandom(32))
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```

```
>>> import signempty
>>> pk,sk = signempty.keypair()
>>> pk.hex()
'61ba682f03259a276dc2d790ed4863113d5559ad7cdd3c282083b9aa6b170ff
>>> sk.hex()
'4645dd39db47dd18b646a34b8f2dc6afd7fa62cc6faafc2ad3426dc94394335
```

Signing and verifying empty messages

```
def sign(message,secret):
  if not isinstance(message,bytes):
    raise TypeError('message must be a byte string')
  if message != b'':
    raise ValueError('message must be empty')
  signedmessage = secret
  return signedmessage

def open(signedmessage,public):
  if len(signedmessage) != 32:
    raise ValueError('bad signature')
  if sha3_256(signedmessage) != public:
    raise ValueError('bad signature')
  message = b''
  return message
```

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  message = b''
  return message
```

```
>>> sm = signempty.sign(b'',sk)
>>> signempty.open(sm,pk)
b''
```

A signature scheme for 1-bit messages: keygen, signing

```
import signempty

def keypair():
  p0,s0 = signempty.keypair()
  p1,s1 = signempty.keypair()
  return (p0,p1),(s0,s1)

def sign(message,secret):
  if not isinstance(message,bytes):
    raise TypeError('message must be a byte string')
  if message == b'0':
    return message,signempty.sign(b'',secret[0])
  if message == b'1':
    return message,signempty.sign(b'',secret[1])
  raise ValueError("message must be b'0' or b'1'")
```

A signature scheme for 1-bit messages: verification

```
def open(signedmessage,public):
  if not isinstance(signedmessage[0],bytes):
    raise TypeError('message must be a byte string')
  if signedmessage[0] == b'0':
    signempty.open(signedmessage[1],public[0])
    return b'0'
  if signedmessage[0] == b'1':
    signempty.open(signedmessage[1],public[1])
    return b'1'
  raise ValueError('bad signature')
```

A signature scheme for 1-bit messages: verification

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def open(signedmessage,public):
  if not isinstance(signedmessage[0],bytes):
    raise TypeError('message must be a byte string')
  if signedmessage[0] == b'0':
    signempty.open(signedmessage[1],public[0])
    return b'0'
  if signedmessage[0] == b'1':
    signempty.open(signedmessage[1],public[1])
    return b'1'
  raise ValueError('bad signature')
```

```
>>> import signbit
>>> pk,sk = signbit.keypair()
>>> sm = signbit.sign(b'1',sk)
>>> signbit.open(sm,pk)
b'1'
```

A signature scheme for 4-bit messages: key generation

```
import signbit

def keypair():
  p0,s0 = signbit.keypair()
  p1,s1 = signbit.keypair()
  p2,s2 = signbit.keypair()
  p3,s3 = signbit.keypair()
  return (p0,p1,p2,p3),(s0,s1,s2,s3)

def sign(m,secret):
  if not isinstance(m,bytes):
    raise TypeError('message must be a byte string')
  if len(m) != 4:
    raise ValueError('message must have length 4')
  sm0 = signbit.sign(m[0:1],secret[0])
  sm1 = signbit.sign(m[1:2],secret[1])
  sm2 = signbit.sign(m[2:3],secret[2])
  sm3 = signbit.sign(m[3:4],secret[3])
  return sm0,sm1,sm2,sm3
```

A signature scheme for 4-bit messages: sign & verify

```
def open(sm,public):
  if len(sm) != 4:
    raise ValueError('signed message must have length 4')
  m0 = signbit.open(sm[0],public[0])
  m1 = signbit.open(sm[1],public[1])
  m2 = signbit.open(sm[2],public[2])
  m3 = signbit.open(sm[3],public[3])
  return m0+m1+m2+m3
```

Do not use one secret key to sign two messages!

```
>>> import sign4bits
>>> pk,sk = sign4bits.keypair()
>>> sm0111 = sign4bits.sign(b'0111',sk)
>>> sign4bits.open(sm0111,pk)
b'0111'
>>> sm1101 = sign4bits.sign(b'1101',sk)
>>> sign4bits.open(sm1101,pk)
b'1101'
```

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>>> import sign4bits
>>> pk,sk = sign4bits.keypair()
>>> sm0111 = sign4bits.sign(b'0111',sk)
>>> sign4bits.open(sm0111,pk)
b'0111'
>>> sm1101 = sign4bits.sign(b'1101',sk)
>>> sign4bits.open(sm1101,pk)
b'1101'

>>> forgery = sm1101[:2]+sm0111[2:]
>>> sign4bits.open(forgery,pk)
b'1111'
```

Lamport's 1-time signature system

Sign arbitrary-length message by signing its 256-bit hash:

```
def hashbits(message):
  h = sha3_256(message)
  return [(b'0',b'1')[1&(h[i//8]>>(i%8))] for i in range(256)]

def keypair():
  keys = [signbit.keypair() for n in range(256)]
  return zip(*keys)

def sign(message,secret):
  hbits = hashbits(message)
  sigs = [signbit.sign(hbits[i],secret[i]) for i in range(256)]
  return sigs,message

def open(sm,public):
  if len(sm[0]) != 256:
    raise ValueError('wrong signature length')
  message = sm[1]
  hbits = hashbits(message)
  for i in range(256):
    if hbits[i] != signbit.open(sm[0][i],public[i]):
      raise ValueError('bit %d of hash does not match'%i)
  return message
```

Can we build shorter signatures?

How big are Lamport's signatures?

- Each Lamport signature has 256 `signbit` signatures.
- Each `signbit` signature has 1 `signempty` signature.
- Each `signempty` signature has one hash output (32 bytes).

Total 256 hash outputs (8192 bytes).

For a 4-bit message: 4 hash outputs (128 bytes).

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Idea for doing better, just 1 hash output for a 4-bit message:

- Define
$$H^i(x) = H(H^{i-1}(x)) = \underbrace{H(H(\cdots(H}_{i \text{ times}}(x)))).$$
- Pick random sk, compute $\text{pk} = H^{16}(\text{sk})$.
- For message $m \in \{0, 1, \ldots, 15\}$ reveal $s = H^m(\text{sk})$ as signature.
- To verify check that $\text{pk} = H^{16-m}(s)$.

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- For message $m \in \{0, 1, \ldots, 15\}$ reveal $s = H^m(\mathrm{sk})$ as signature.
- To verify check that $\mathrm{pk} = H^{16-m}(s)$.

This is the weak Winternitz signature system.

Weak Winternitz

```
def keypair():
  secret = sha3_256(os.urandom(32))
  public = secret
  for i in range(16): public = sha3_256(public)
  return public,secret

def sign(m,secret):
  if not isinstance(m,int) or m<0 or m>15:
    raise ValueError('message must be in {0,1,...,15}')
  s = secret
  for i in range(m): s = sha3_256(s)
  return s,m

def open(sm,public):
  if not isinstance(sm[1],int) or sm[1]<0 or sm[1]>15:
    raise ValueError('message must be in {0,1,...,15}')
  c = sm[0]
  for i in range(16-sm[1]): c = sha3_256(c)
  if c != public: raise ValueError('bad signature')
  return sm[1]
```

Why this is “weak” Winternitz

This is insecure even if you sign only 1 message!

```
>>> import weak_winternitz
>>> pk,sk = weak_winternitz.keypair()
>>> sm7 = weak_winternitz.sign(7,sk)
>>> H = weak_winternitz.sha3_256
>>> weak_winternitz.open(sm7,pk)
7
>>> forgery = H(sm7[0]),8
>>> weak_winternitz.open(forgery,pk)
8
>>> forgery2 = H(forgery[0]),9
>>> weak_winternitz.open(forgery2,pk)
9
>>>
```

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9
>>>
```

Fix: Strong Winternitz uses weak Winternitz twice,
running one chain forward, one chain backward.
(Exercise: this is safe with H^{15} instead of H^{16} in weak Winternitz.)

Strong Winternitz

```
import weak_winternitz

def keypair():
  keys = [weak_winternitz.keypair() for n in range(2)]
  return zip(*keys)

def sign(m,secret):
  if not isinstance(m,int) or m<0 or m>15:
    raise ValueError('message must be in {0,1,...,15}')
  sign0 = weak_winternitz.sign(m,secret[0])
  sign1 = weak_winternitz.sign(15-m,secret[1])
  return sign0[0],sign1[0],m

def open(sm,public):
  if not isinstance(sm[2],int) or sm[2]<0 or sm[2]>15:
    raise ValueError('message must be in {0,1,...,15}')
  weak_winternitz.open((sm[0],sm[2]),public[0])
  weak_winternitz.open((sm[1],15-sm[2]),public[1])
  return sm[2]
```

The complete Winternitz system

Define parameter w. Each chain will run for 2^w steps.

For signing a 256-bit hash this needs $t_1 = \lceil 256/w \rceil$ chains.
Write m in base 2^w (integers of w bits):

$$m = (m_{t_1-1}, \ldots, m_1, m_0)$$

(zero–padding if necessary).

Put

$$c = \sum_{i=0}^{t_1-1} (2^w - m_i)$$

Note that $c \leq t_1 2^w$.

The checksum c gets larger if m_i is smaller.

Write c in base 2^w. This takes $t_2 = 1 + \lceil \lfloor (\log_2 t_1 \rfloor + 1)/w \rceil$ w-bit integers

$$c = (c_{t_2-1}, \ldots, c_1, c_0).$$

Publish $t_1 + t_2$ public keys, sign with chains of lengths

$$m_{t_1-1}, \ldots, m_1, m_0, c_{t_2-1}, \ldots, c_1, c_0.$$

The complete Winternitz system for $w = 8$

Define parameter $w = 8$. Each chain will run for $2^8 = 256$ steps.

For signing a 256-bit hash this needs $t_1 = \lceil 256/8 \rceil = 32$ chains.
Write m in base 2^8 (integers of 8 bits):

$$m = (m_{31}, \ldots, m_1, m_0)$$

(zero–padding if necessary).

Put

$$c = \sum_{i=0}^{31} (2^8 - m_i)$$

Note that $c \leq 32 \cdot 2^8 = 2^{13}$.

The checksum c gets larger if m_i is smaller.

Write c in base 2^8. This takes $t_2 = 1 + \lceil (5+1)/8 \rceil = 2$
8-bit integers

$$c = (c_1, c_0).$$

Publish $t_1 + t_2 = 34$ public keys, sign with chains of lengths

$$m_{31}, \ldots, m_1, m_0, c_1, c_0.$$

Exercise

How does Winternitz with $w = 5$ compare to Winternitz with $w = 8$ for signing a 256-bit hash? Efficiency metrics:

- How many bytes are in the signature?
- How many bytes are in the public key?
- How many bytes are in the secret key?
- How many hash-function computations are needed in signing?
- How many hash-function computations are needed in verifying?

Remember that you also need to sign the checksum component!