Quantum algorithms

Daniel J. Bernstein

"Quantum algorithm"
means an algorithm that
a quantum computer can run.

i.e. a sequence of instructions,
where each instruction is
in a quantum computer's
supported instruction set.

How do we know which instructions a quantum computer will support?

Quantum computer type 1 (QC1):
contains many “qubits”;
can efficiently perform
“NOT gate”, “Hadamard gate”,
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to compute "Toffoli gate";
... "Simon's algorithm";
... "Shor's algorithm"; etc.

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Combine these instructions to compute “Toffoli gate”; . . . “Simon’s algorithm”; . . . “Shor’s algorithm”; etc.

General belief: Traditional CPU isn’t QC1; e.g. can’t factor quickly.

Quantum computer type 2 (QC2): stores a simulated universe; efficiently simulates the laws of quantum physics with as much accuracy as desired.

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General belief: any QC1 is a QC2. Partial proof: see, e.g., 2011 Jordan–Lee–Preskill "Quantum algorithms for quantum field theories".

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Argument for belief: any physical computer must follow the laws of quantum physics, so a QC2 can efficiently simulate any physical computer.

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General belief: any QC3 is a QC1.
Argument for belief:
look, we're building a QC1.

A note on D-Wave

Apparent scientific consensus:
Current "quantum computers"
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Is D-Wave a bad investment?

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a list of 3 elements of $\{0, 1\}$.
e.g.: $(0, 0, 0)$.

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Data stored in 4 qubits: a list of 16 numbers, not all zero. e.g.:
$(3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9, 3)$.

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Data stored in 64 qubits:
a list of 2^{64} numbers, not all zero.

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Data stored in 64 qubits:
a list of 2^{64} numbers, not all zero.

Data stored in 1000 qubits: a list of 2^{1000} numbers, not all zero.

Measuring a quantum computer

Can simply look at a bit.
Cannot simply look at the list
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State is then all zeros except 1 at position q.

e.g.: Say 3 qubits have state $(1, 1, 1, 1, 1, 1, 1, 1)$.

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Measurement produces
$000 = 0$ with probability $1/8$;
$001 = 1$ with probability $1/8$;
$010 = 2$ with probability $1/8$;
$011 = 3$ with probability $1/8$;
$100 = 4$ with probability $1/8$;
$101 = 5$ with probability $1/8$;
$110 = 6$ with probability $1/8$;
$111 = 7$ with probability $1/8$.

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"Quantum RNG."

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"Quantum RNG."

Warning: Quantum RNGs sold today are measurably biased.

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Measurement produces
$000 = 0$ with probability $9/173$;
$001 = 1$ with probability $1/173$;
$010 = 2$ with probability $16/173$;
$011 = 3$ with probability $1/173$;
$100 = 4$ with probability $25/173$;
$101 = 5$ with probability $81/173$;
$110 = 6$ with probability $4/173$;
$111 = 7$ with probability $36/173$.

e.g.: Say 3 qubits have state $(3, 1, 4, 1, 5, 9, 2, 6)$.

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$000 = 0$ with probability $9/173$;
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$010 = 2$ with probability $16/173$;
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$100 = 4$ with probability $25/173$;
$101 = 5$ with probability $81/173$;
$110 = 6$ with probability $4/173$;
$111 = 7$ with probability $36/173$.

5 is most likely outcome.

e.g.: Say 3 qubits have state

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$000 = 0$ with probability 0;
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$101 = 5$ with probability 1;
$110 = 6$ with probability 0;
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e.g.: Say 3 qubits have state $(0, 0, 0, 0, 0, 1, 0, 0)$.

Measurement produces

$000 = 0$ with probability 0;

$001 = 1$ with probability 0;

$010 = 2$ with probability 0;

$011 = 3$ with probability 0;

$100 = 4$ with probability 0;

$101 = 5$ with probability 1;

$110 = 6$ with probability 0;

$111 = 7$ with probability 0.

5 is guaranteed outcome.

NOT gates

NOT_0 gate on 3 qubits:

$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$

$(1, 3, 1, 4, 9, 5, 6, 2)$.

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NOT_0 gate on 4 qubits:
$(3,1,4,1,5,9,2,6,5,3,5,8,9,7,9,3) \mapsto$
$(1,3,1,4,9,5,6,2,3,5,8,5,7,9,3,9)$.

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NOT_1 gate on 3 qubits:
$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$
$(4, 1, 3, 1, 2, 6, 5, 9)$.

NOT gates

NOT_0 gate on 3 qubits:
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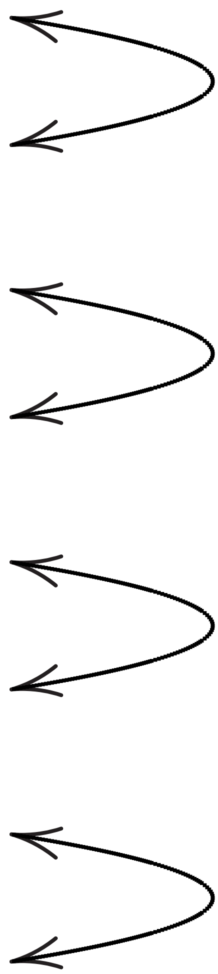
$(1, 3, 1, 4, 9, 5, 6, 2)$.

NOT_0 gate on 4 qubits:
$(3,1,4,1,5,9,2,6,5,3,5,8,9,7,9,3) \mapsto$
$(1,3,1,4,9,5,6,2,3,5,8,5,7,9,3,9)$.

NOT_1 gate on 3 qubits:
$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$
$(4, 1, 3, 1, 2, 6, 5, 9)$.

NOT_2 gate on 3 qubits:
$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$
$(5, 9, 2, 6, 3, 1, 4, 1)$.

state	measurement
$(1,0,0,0,0,0,0,0)$	000
$(0,1,0,0,0,0,0,0)$	001
$(0,0,1,0,0,0,0,0)$	010
$(0,0,0,1,0,0,0,0)$	011
$(0,0,0,0,1,0,0,0)$	100
$(0,0,0,0,0,1,0,0)$	101
$(0,0,0,0,0,0,1,0)$	110
$(0,0,0,0,0,0,0,1)$	111



Operation on quantum state:
NOT_0, swapping pairs.
Operation after measurement:
flipping bit 0 of result.
Flip: output is not input.

Controlled-NOT (CNOT) gates

e.g. C_1NOT_0:

$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$

$(3, 1, 1, 4, 5, 9, 6, 2)$.

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flipping bit 0 *if* bit 1 is set; i.e.,
$(q_2, q_1, q_0) \mapsto (q_2, q_1, q_0 \oplus q_1)$.

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e.g. C_0NOT_2:
$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$
$(3, 9, 4, 6, 5, 1, 2, 1)$.

Toffoli gates

Also known as CCNOT gates: controlled-controlled-NOT gates.

e.g. $C_2C_1NOT_0$:
$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$
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e.g. $C_0C_1NOT_2$:
$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$
$(3, 1, 4, 6, 5, 9, 2, 1)$.

More shuffling

Combine NOT, CNOT, Toffoli to build other permutations.

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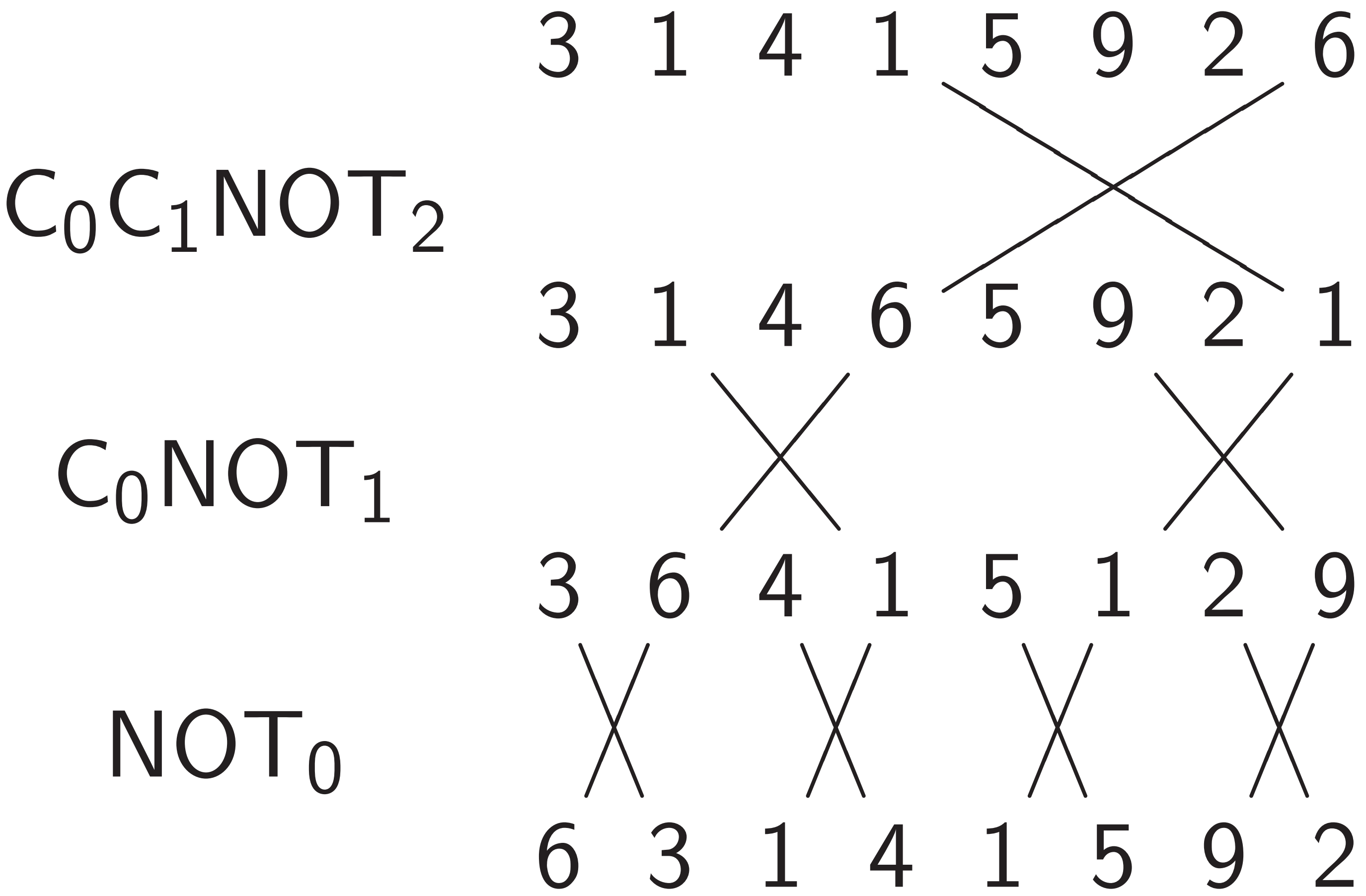
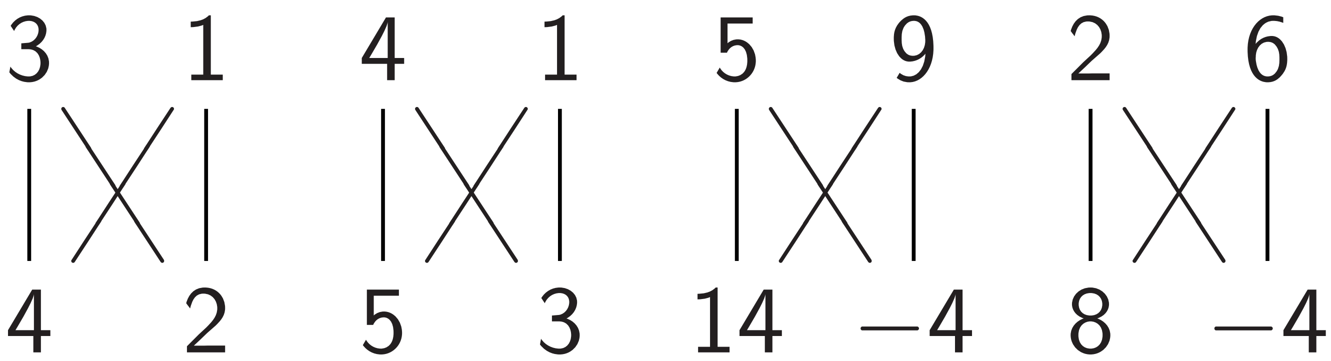
e.g. series of gates to rotate 8 positions by distance 1:



Hadamard gates

Hadamard$_0$:

$(a, b) \mapsto (a + b, a - b)$.



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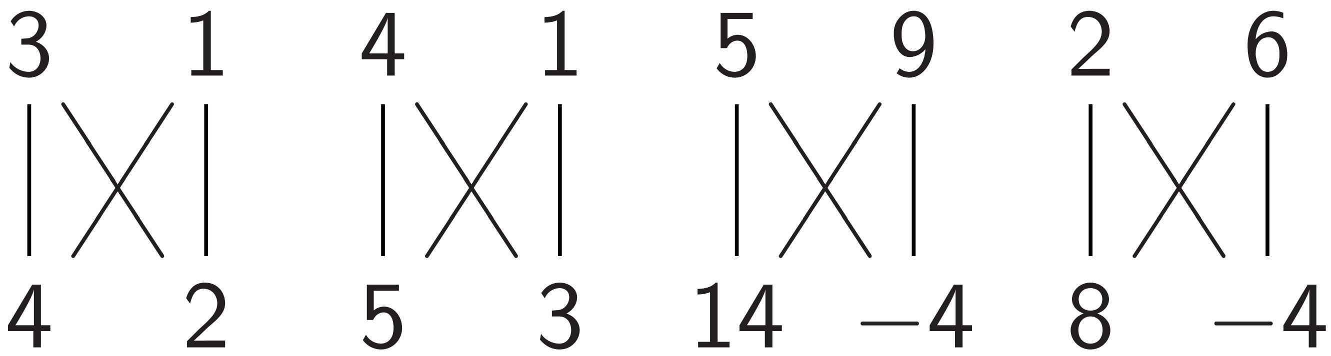
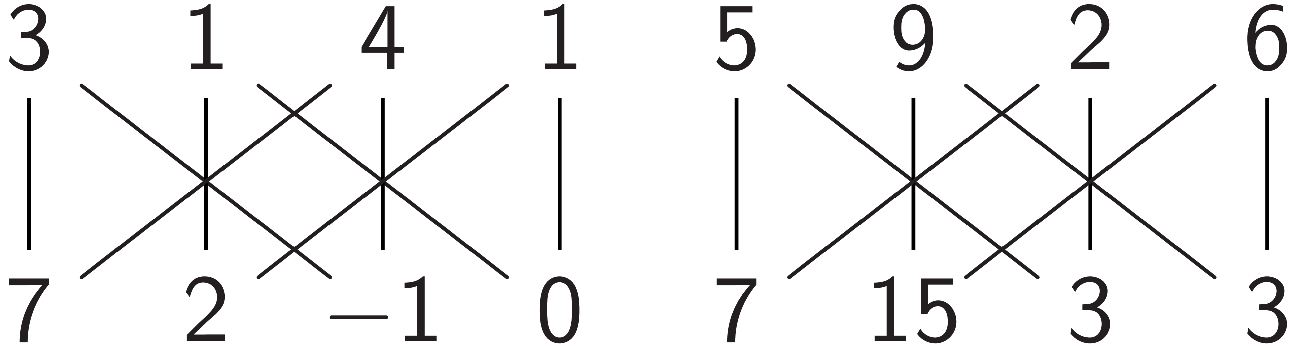


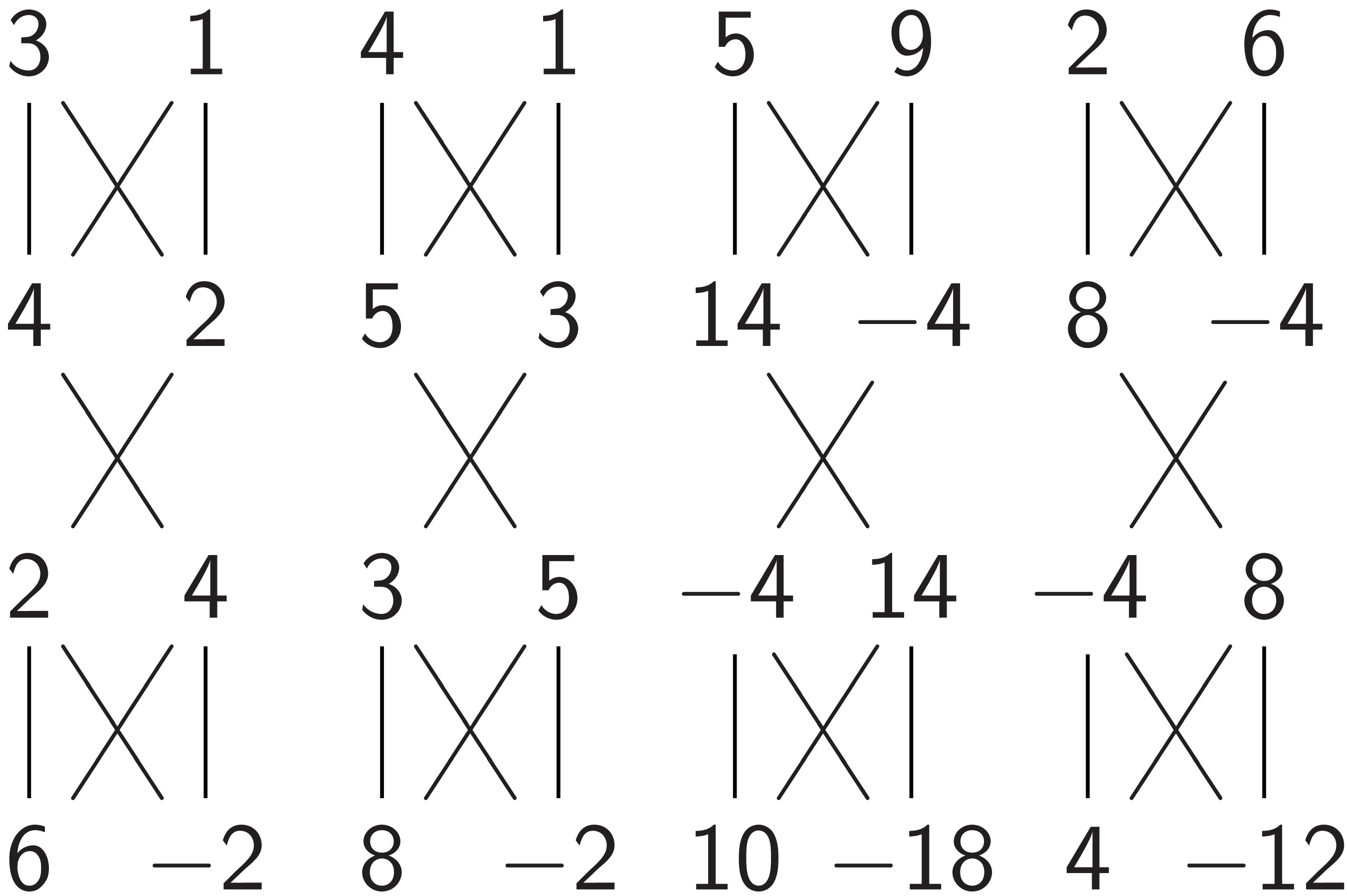
Hadamard$_1$:

$(a, b, c, d) \mapsto$
$(a + c, b + d, a - c, b - d)$.



Some uses of Hadamard gates

Hadamard_0, NOT_0, Hadamard_0:



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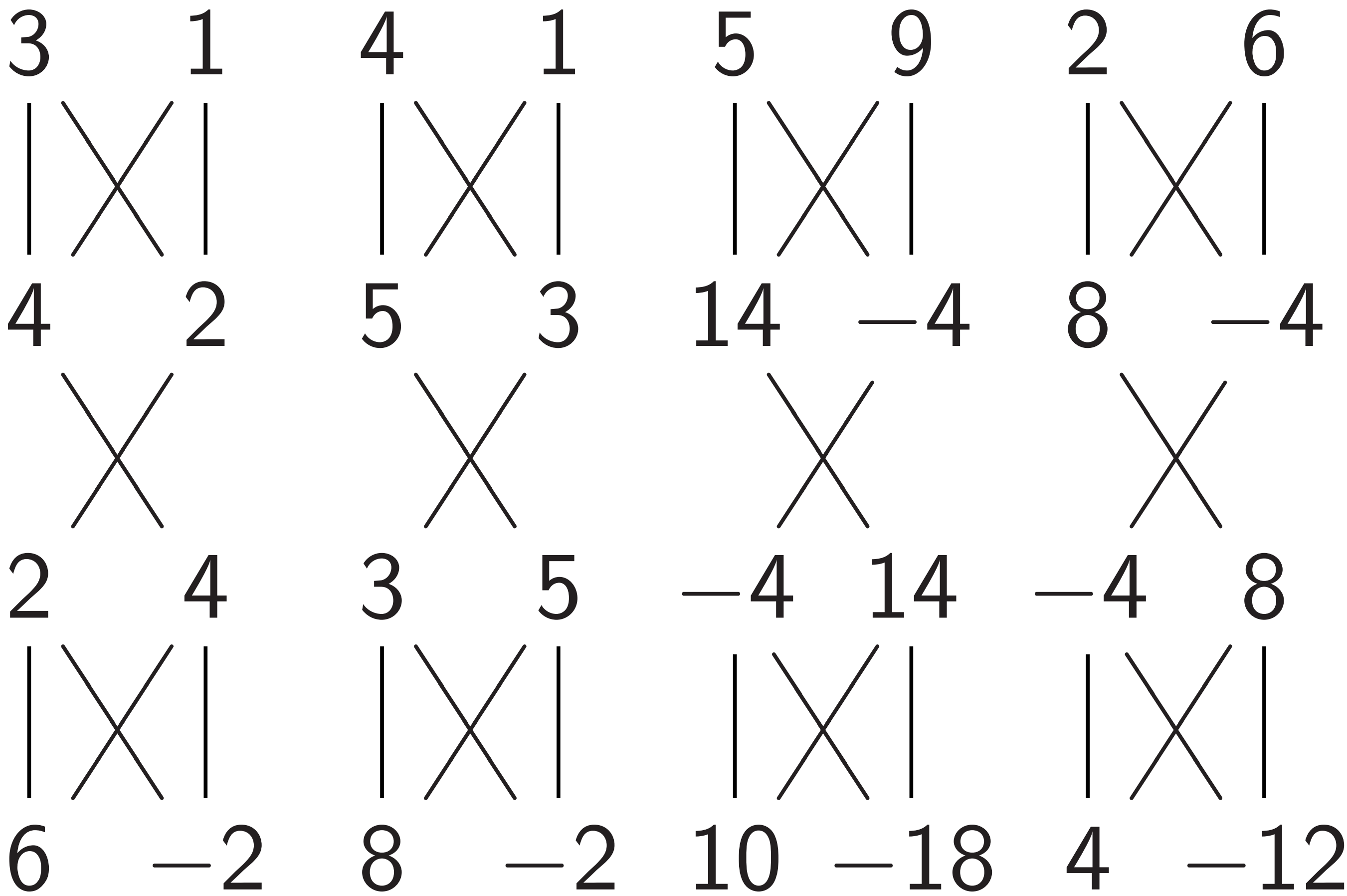

"Multiply each amplitude by 2."

This is not physically observable.

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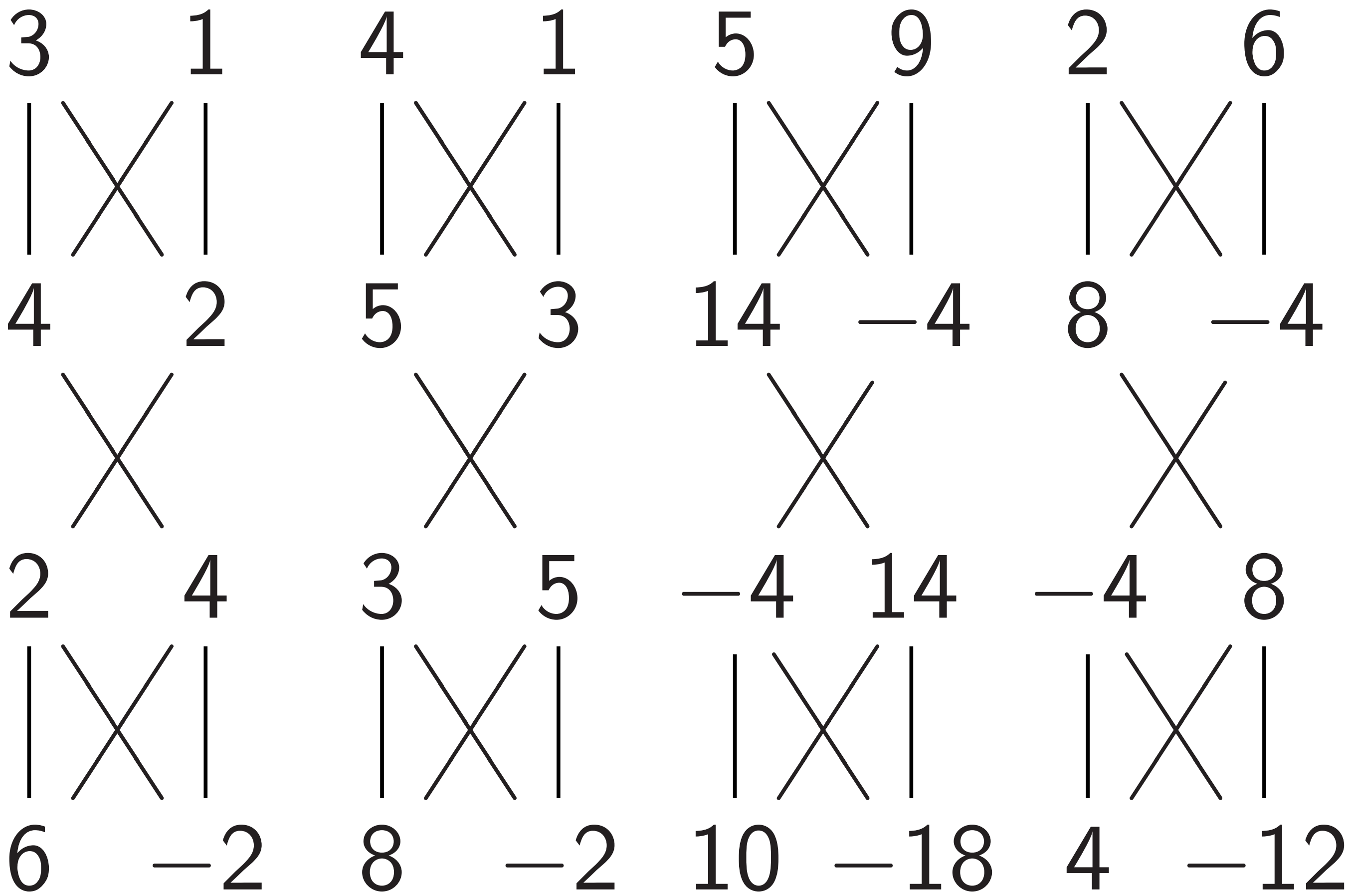


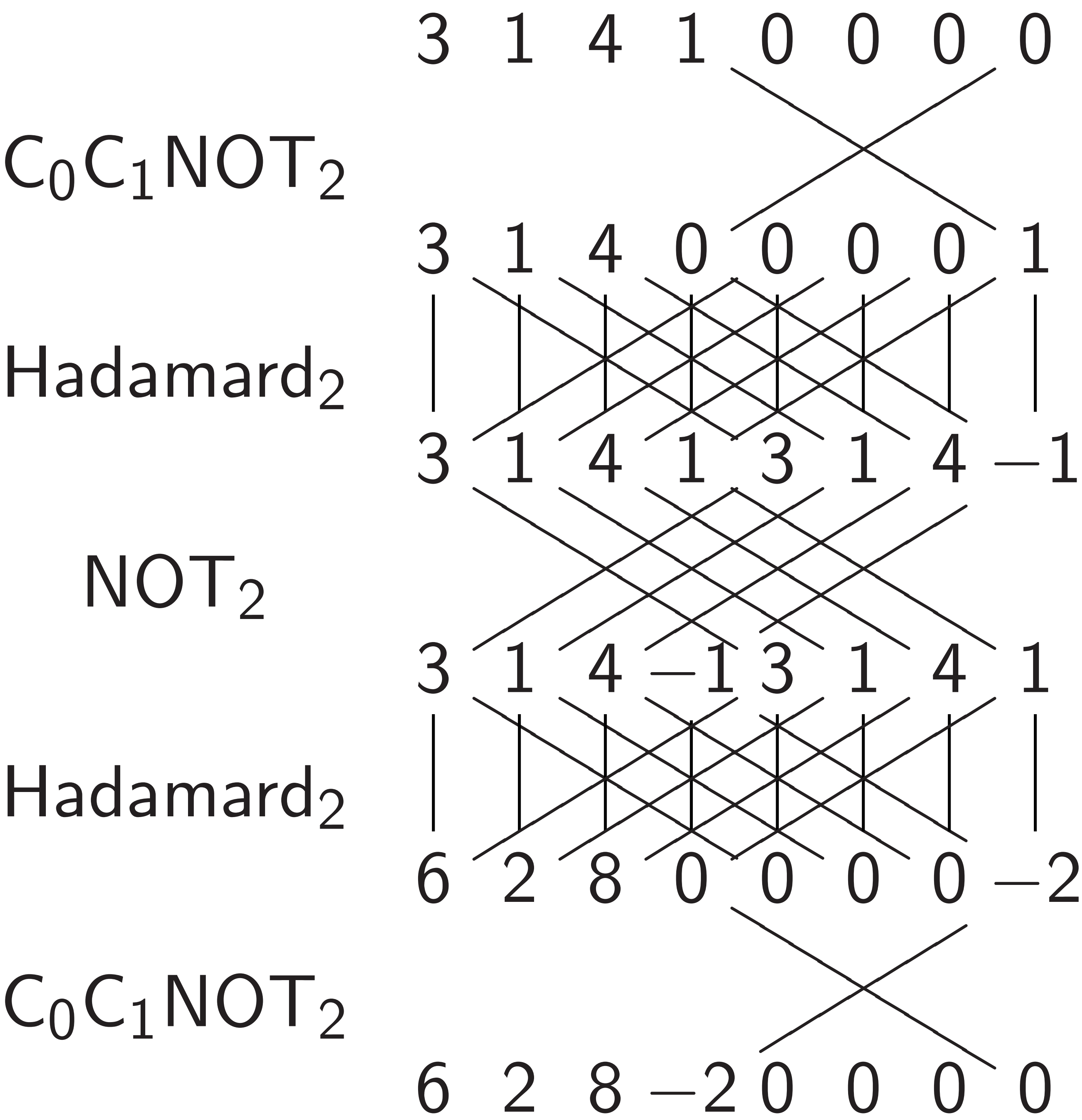
"Multiply each amplitude by 2."
This is not physically observable.

"Negate amplitude if q_0 is set."
No effect on measuring *now*.

Fancier example:

"Negate amplitude if $q_0 q_1$ is set."

Assumes $q_2 = 0$: "ancilla" qubit.



Affects measurements: "Negate amplitude around its average."

$(3, 1, 4, 1) \mapsto (1.5, 3.5, 0.5, 3.5)$.

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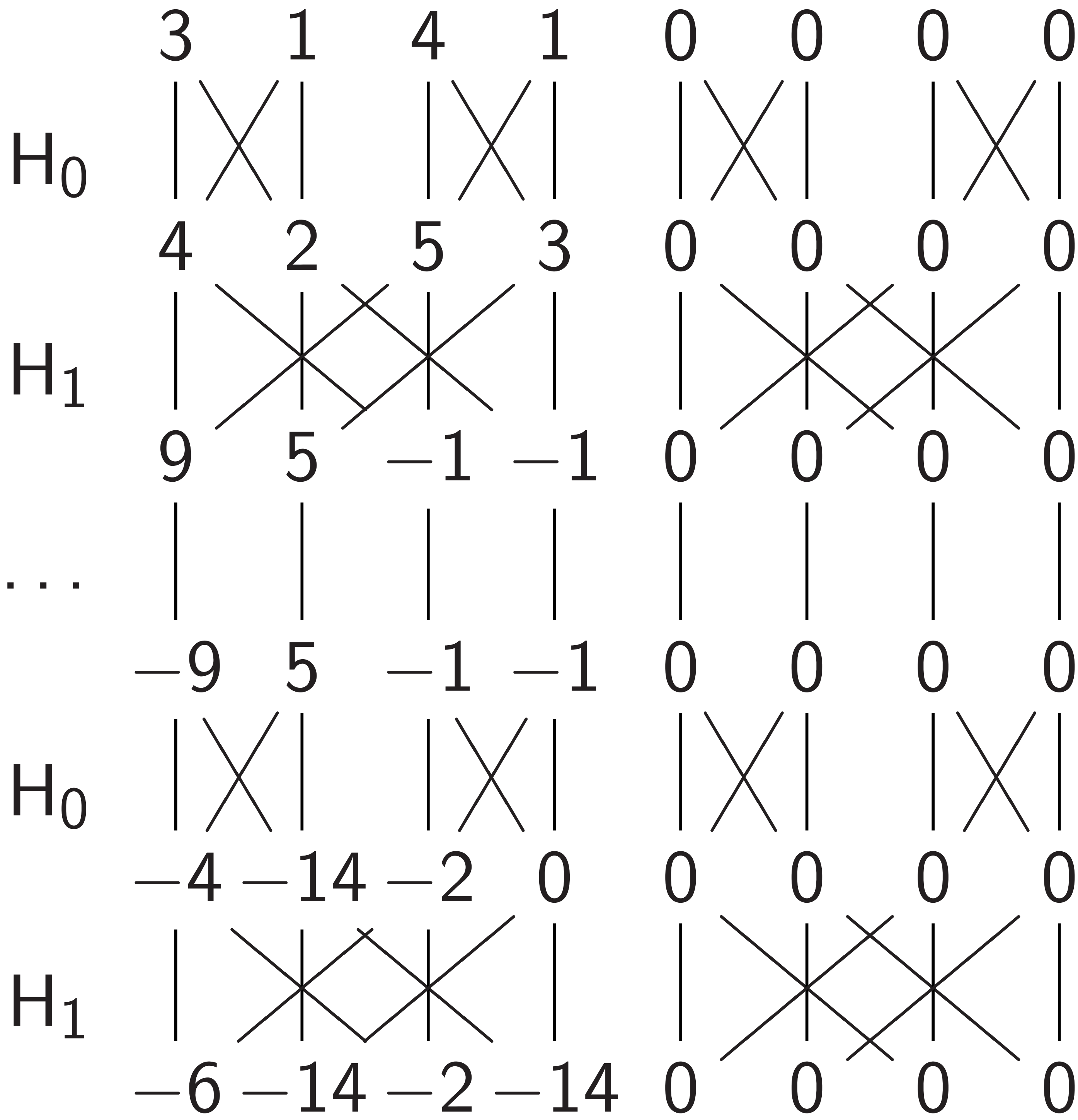

Simon's algorithm

Assumptions:

- Given any $u \in \{0,1\}^n$, can efficiently compute $f(u)$.
- Nonzero $s \in \{0,1\}^n$.
- $f(u) = f(u \oplus s)$ for all u.
- f has no other collisions.

Goal: Figure out s.

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Traditional algorithm to find s: compute f for many inputs, hope to find collision.

Simon's algorithm finds s with $\approx n$ quantum computations of f.

Example of Simon's algorithm

Step 1. Set up pure zero state:

1, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0.

Example of Simon's algorithm

Step 2. Hadamard$_0$:

1, 1, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0.

Example of Simon's algorithm

Step 3. Hadamard$_1$:

1, 1, 1, 1, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0.

Example of Simon's algorithm

Step 4. Hadamard_2:

1, 1, 1, 1, 1, 1, 1, 1,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0.

Each column is a parallel universe.

Example of Simon's algorithm

Step 5. C_0NOT_3:

1, 0, 1, 0, 1, 0, 1, 0,
0, 1, 0, 1, 0, 1, 0, 1,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0.

Each column is a parallel universe performing its own computations.

Example of Simon's algorithm

Step 5b. More shuffling:

1, 0, 0, 0, 1, 0, 0, 0,
0, 1, 0, 0, 0, 1, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 1, 0, 0, 0, 1, 0,
0, 0, 0, 1, 0, 0, 0, 1,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0.

Each column is a parallel universe performing its own computations.

Example of Simon's algorithm

Step 5c. More shuffling:

1, 0, 0, 0, 0, 0, 0, 0,
0, 1, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 1, 0, 0, 0,
0, 0, 0, 0, 0, 1, 0, 0,
0, 0, 1, 0, 0, 0, 0, 0,
0, 0, 0, 1, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 1, 0,
0, 0, 0, 0, 0, 0, 0, 1.

Each column is a parallel universe performing its own computations.

Example of Simon's algorithm

Step 5d. More shuffling:

1, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 1, 0, 0,

0, 0, 0, 0, 1, 0, 0, 0,

0, 1, 0, 0, 0, 0, 0, 0,

0, 0, 1, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 1,

0, 0, 0, 0, 0, 0, 1, 0,

0, 0, 0, 1, 0, 0, 0, 0.

Each column is a parallel universe performing its own computations.

Example of Simon's algorithm

Step 5e. More shuffling:

1, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 1, 0, 0,
0, 0, 0, 0, 1, 0, 0, 0,
0, 1, 0, 0, 0, 0, 0, 0,
0, 0, 1, 0, 0, 0, 0, 1,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 1, 0, 0, 1, 0,
0, 0, 0, 0, 0, 0, 0, 0.

Each column is a parallel universe performing its own computations.

Example of Simon's algorithm

Step 5f. More shuffling:

0, 0, 0, 0, 0, 1, 0, 0,
1, 0, 0, 0, 0, 0, 0, 0,
0, 1, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 1, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 1, 0, 0, 0, 0, 1,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 1, 0, 0, 1, 0.

Each column is a parallel universe performing its own computations.

Example of Simon's algorithm

Step 5g. More shuffling:

0, 1, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 1, 0, 0, 0,
0, 0, 0, 0, 0, 1, 0, 0,
1, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 1, 0, 0, 1, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 1, 0, 0, 0, 0, 1.

Each column is a parallel universe performing its own computations.

Example of Simon's algorithm

Step 5h. More shuffling:

0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 1, 0, 0, 1, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 1, 0, 0, 0, 0, 1,
0, 1, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 1, 0, 0, 0,
0, 0, 0, 0, 0, 1, 0, 0,
1, 0, 0, 0, 0, 0, 0, 0.

Each column is a parallel universe performing its own computations.

Example of Simon's algorithm

Step 5i. More shuffling:

0, 0, 0, 0, 0, 0, 1, 0,

0, 0, 0, 1, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 1,

0, 0, 1, 0, 0, 0, 0, 0,

0, 1, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 1, 0, 0, 0,

0, 0, 0, 0, 0, 1, 0, 0,

1, 0, 0, 0, 0, 0, 0, 0.

Each column is a parallel universe performing its own computations.

Example of Simon's algorithm

Step 5j. Final shuffling:

0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 1, 0, 0, 1, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 1, 0, 0, 0, 0, 1,
0, 1, 0, 0, 1, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 0, 1, 0, 0.

Each column is a parallel universe performing its own computations.

Example of Simon's algorithm

Step 5j. Final shuffling:

0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 1, 0, 0, 1, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 1, 0, 0, 0, 0, 1,
0, 1, 0, 0, 1, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 0, 1, 0, 0.

Each column is a parallel universe performing its own computations.

Surprise: u and $u \oplus 101$ match.

Example of Simon's algorithm

Step 6. Hadamard$_0$:

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 1, $\overline{1}$, 0, 0, 1, 1,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 1, 1, 0, 0, 1, $\overline{1}$,

1, $\overline{1}$, 0, 0, 1, 1, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

1, 1, 0, 0, 1, $\overline{1}$, 0, 0.

Notation: $\overline{1}$ means -1.

Example of Simon's algorithm

Step 7. Hadamard$_1$:

$0, 0, 0, 0, 0, 0, 0, 0,$

$1, \bar{1}, \bar{1}, 1, 1, 1, \bar{1}, \bar{1},$

$0, 0, 0, 0, 0, 0, 0, 0,$

$1, 1, \bar{1}, \bar{1}, 1, \bar{1}, \bar{1}, 1,$

$1, \bar{1}, 1, \bar{1}, 1, 1, 1, 1,$

$0, 0, 0, 0, 0, 0, 0, 0,$

$0, 0, 0, 0, 0, 0, 0, 0,$

$1, 1, 1, 1, 1, \bar{1}, 1, \bar{1}.$

Example of Simon's algorithm

Step 8. Hadamard$_2$:

0, 0, 0, 0, 0, 0, 0, 0,

2, 0, $\overline{2}$, 0, 0, $\overline{2}$, 0, 2,

0, 0, 0, 0, 0, 0, 0, 0,

2, 0, $\overline{2}$, 0, 0, 2, 0, $\overline{2}$,

2, 0, 2, 0, 0, $\overline{2}$, 0, $\overline{2}$,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

2, 0, 2, 0, 0, 2, 0, 2.

Example of Simon's algorithm

Step 8. Hadamard$_2$:

0, 0, 0, 0, 0, 0, 0, 0,

2, 0, $\overline{2}$, 0, 0, $\overline{2}$, 0, 2,

0, 0, 0, 0, 0, 0, 0, 0,

2, 0, $\overline{2}$, 0, 0, 2, 0, $\overline{2}$,

2, 0, 2, 0, 0, $\overline{2}$, 0, $\overline{2}$,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

2, 0, 2, 0, 0, 2, 0, 2.

Step 9: Measure. Obtain some information about the surprise: a random vector orthogonal to 101.

Repeat to figure out 101.

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Generalize Step 5 to any function $u \mapsto f(u)$ with $f(u) = f(u \oplus s)$. "Usually" algorithm figures out s.

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e.g. Shor finds "random" s with $2^u \bmod N = 2^{u+s} \bmod N$. Easy to factor N using this.

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Many spectacular applications.

e.g. Shor finds "random" s with $2^u \bmod N = 2^{u+s} \bmod N$.
Easy to factor N using this.

e.g. Shor finds "random" s, t with $4^u 9^v \bmod p = 4^{u+s} 9^{v+t} \bmod p$.
Easy to compute discrete logs.

Grover's algorithm

Assume: unique $s \in \{0,1\}^n$ has $f(s) = 0$.

Traditional algorithm to find s: compute f for many inputs, hope to find output 0. Success probability is very low until #inputs approaches 2^n.

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has $f(s) = 0$.

Traditional algorithm to find s:
compute f for many inputs,
hope to find output 0.
Success probability is very low
until #inputs approaches 2^n.

Grover's algorithm takes only $2^{n/2}$
reversible computations of f.
Typically: reversibility overhead
is small enough that this
easily beats traditional algorithm.

Start from uniform superposition over all n-bit strings u.

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Step 1: Set $a \leftarrow b$ where
$b_u = -a_u$ if $f(u) = 0$,
$b_u = a_u$ otherwise.
This is fast.

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Negate a around its average.
This is also fast.

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Repeat Step 1 + Step 2 about $0.58 \cdot 2^{0.5n}$ times.

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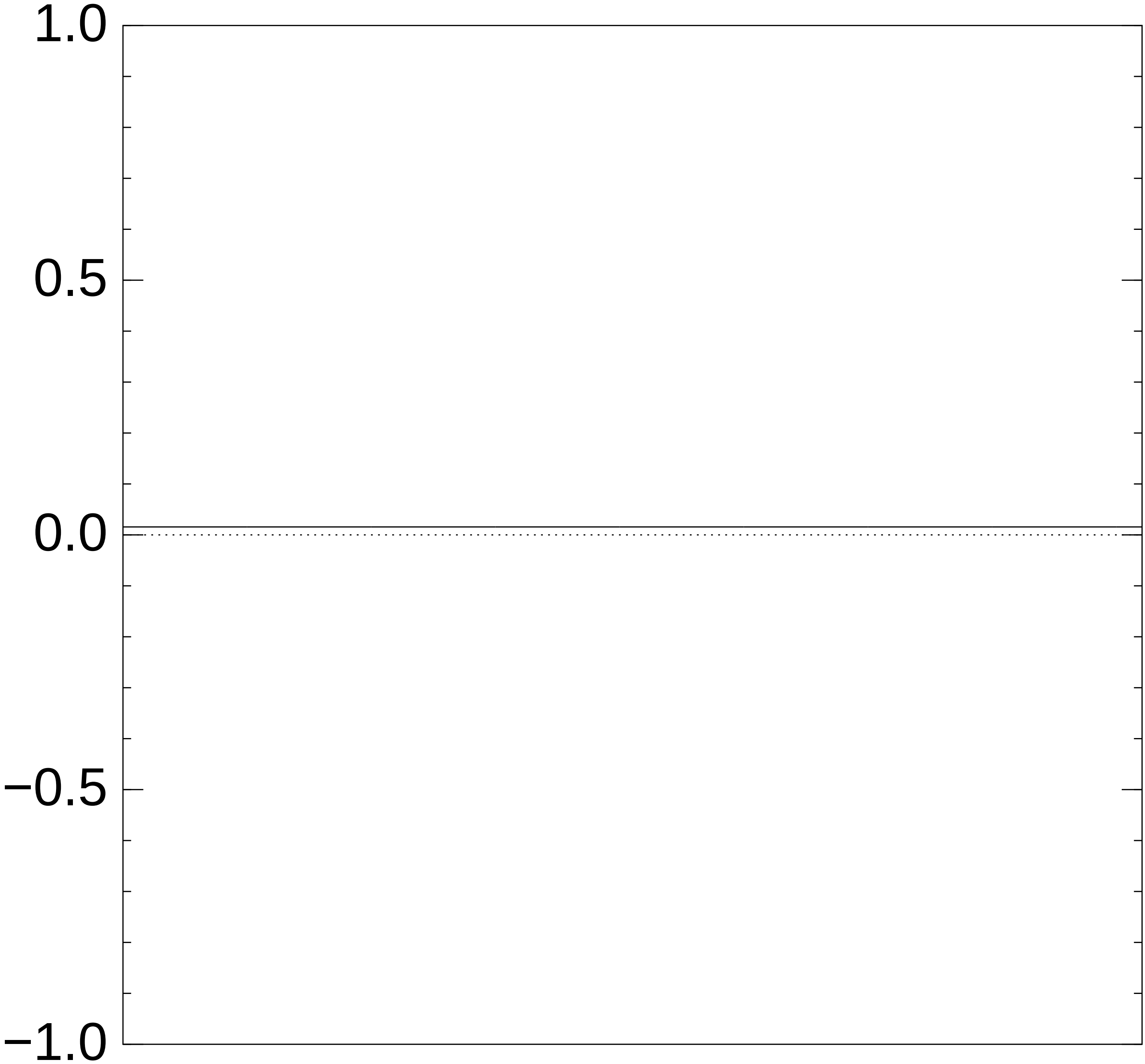
Step 2: "Grover diffusion".
Negate a around its average.
This is also fast.

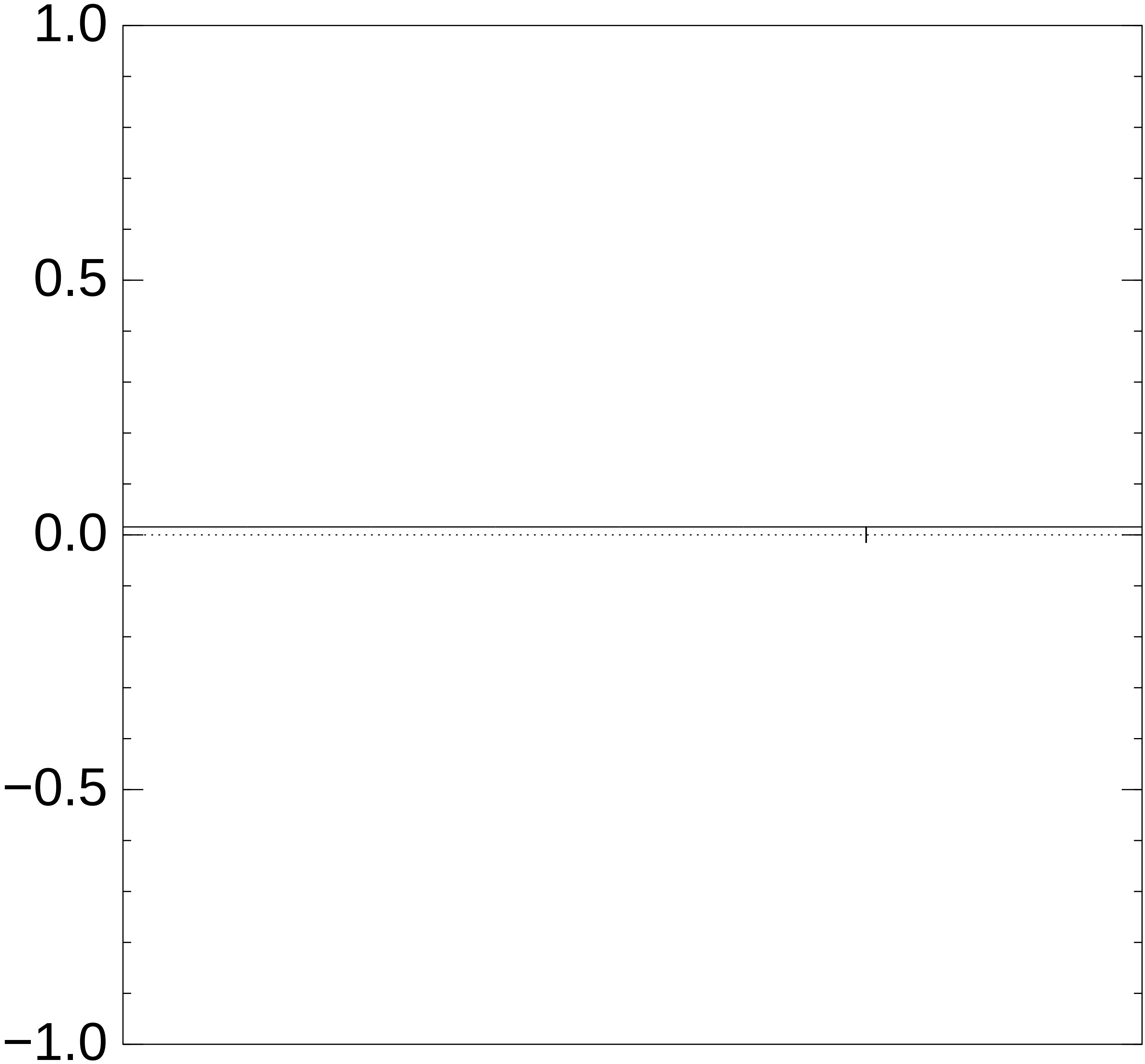
Repeat Step 1 + Step 2 about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds s.

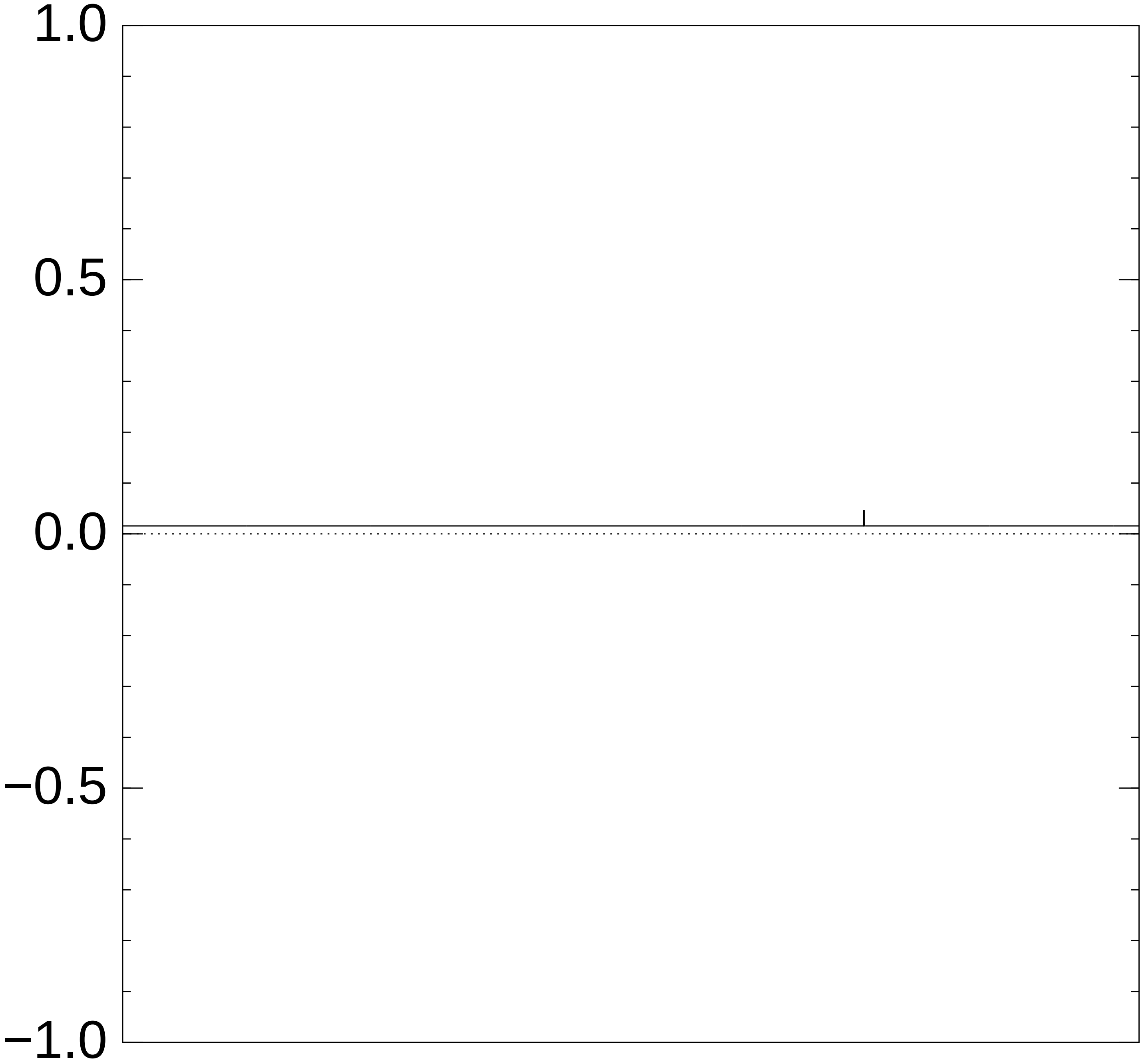
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after 0 steps:



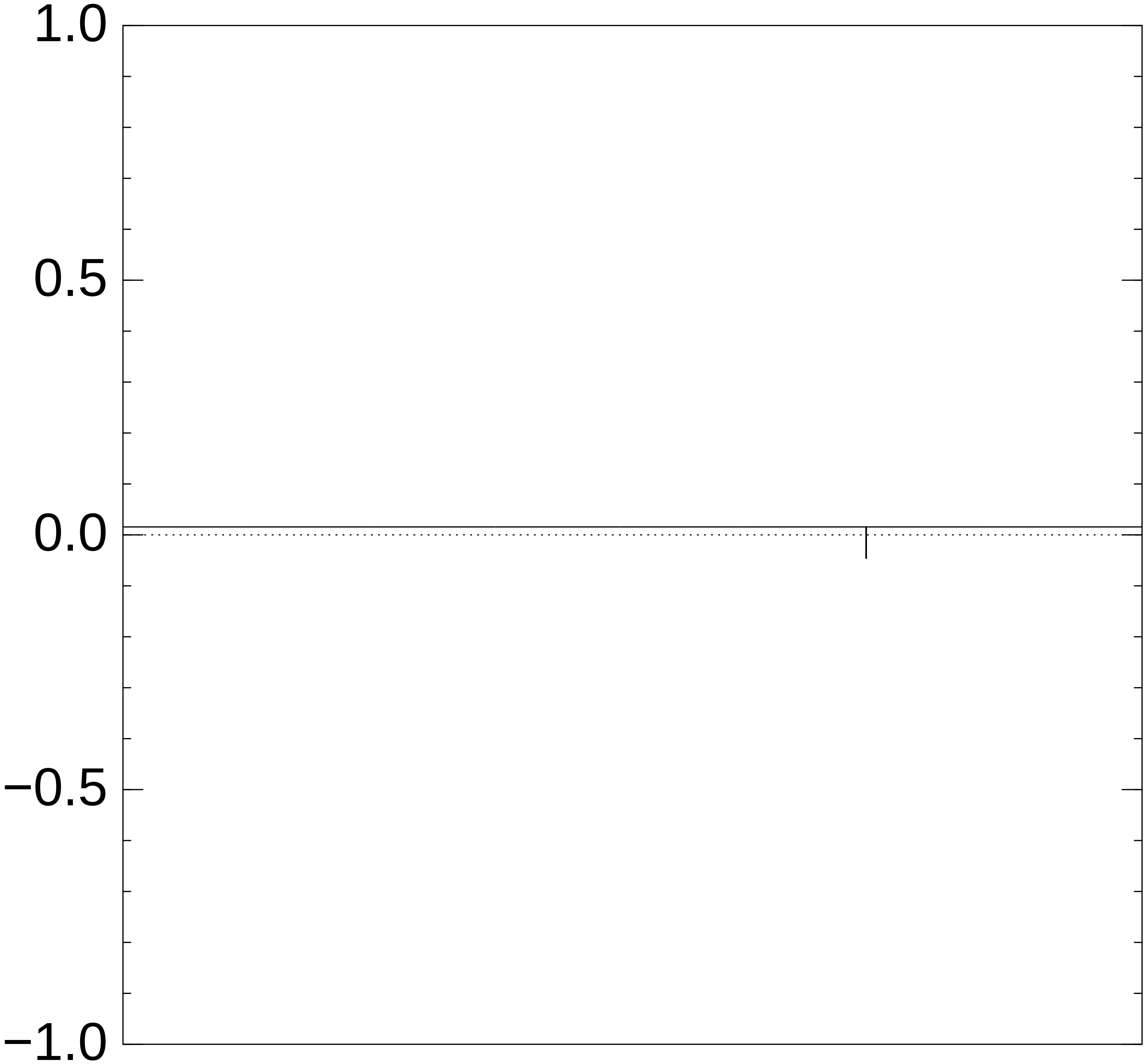
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after Step 1:



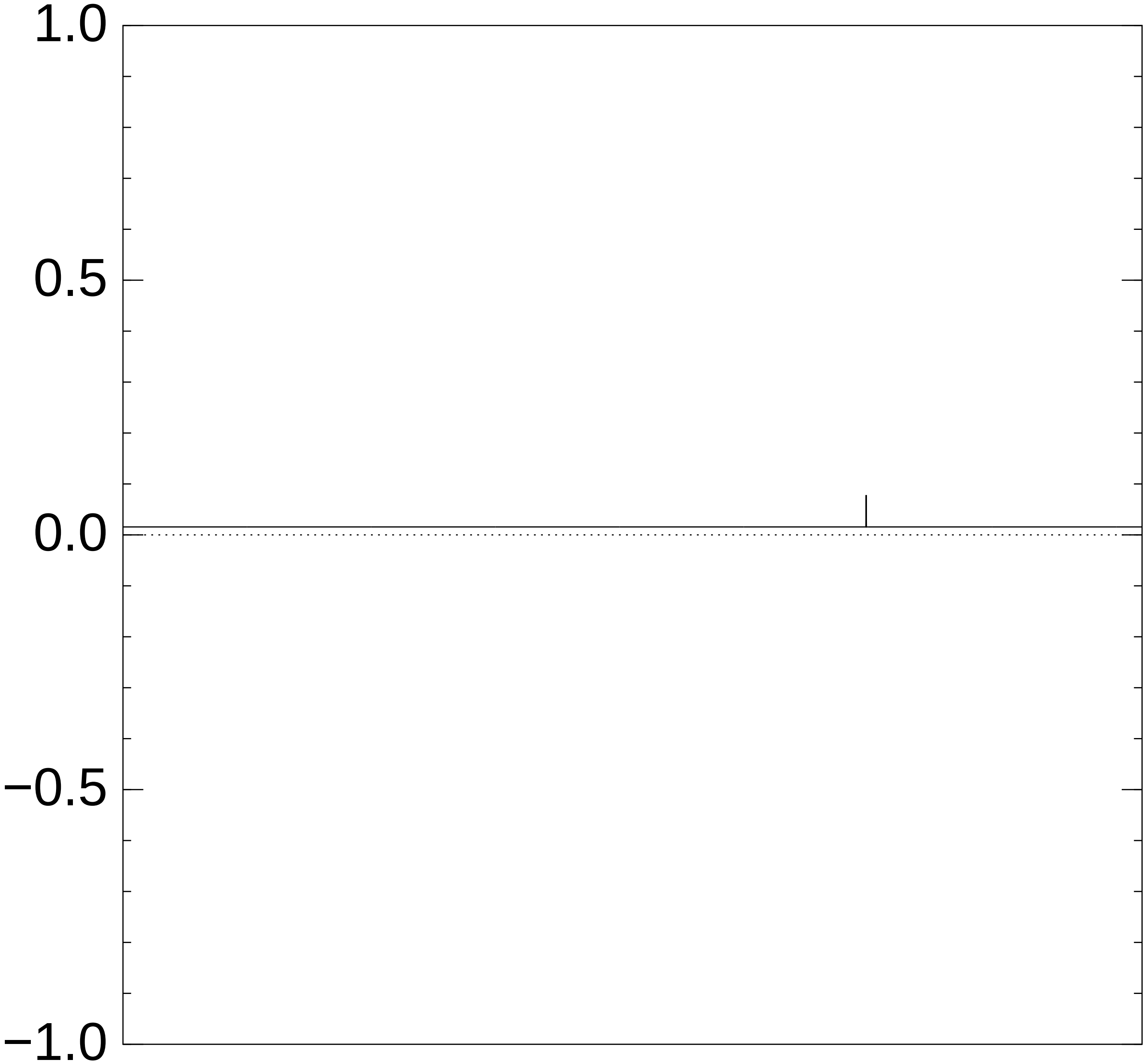
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after Step 1 + Step 2:



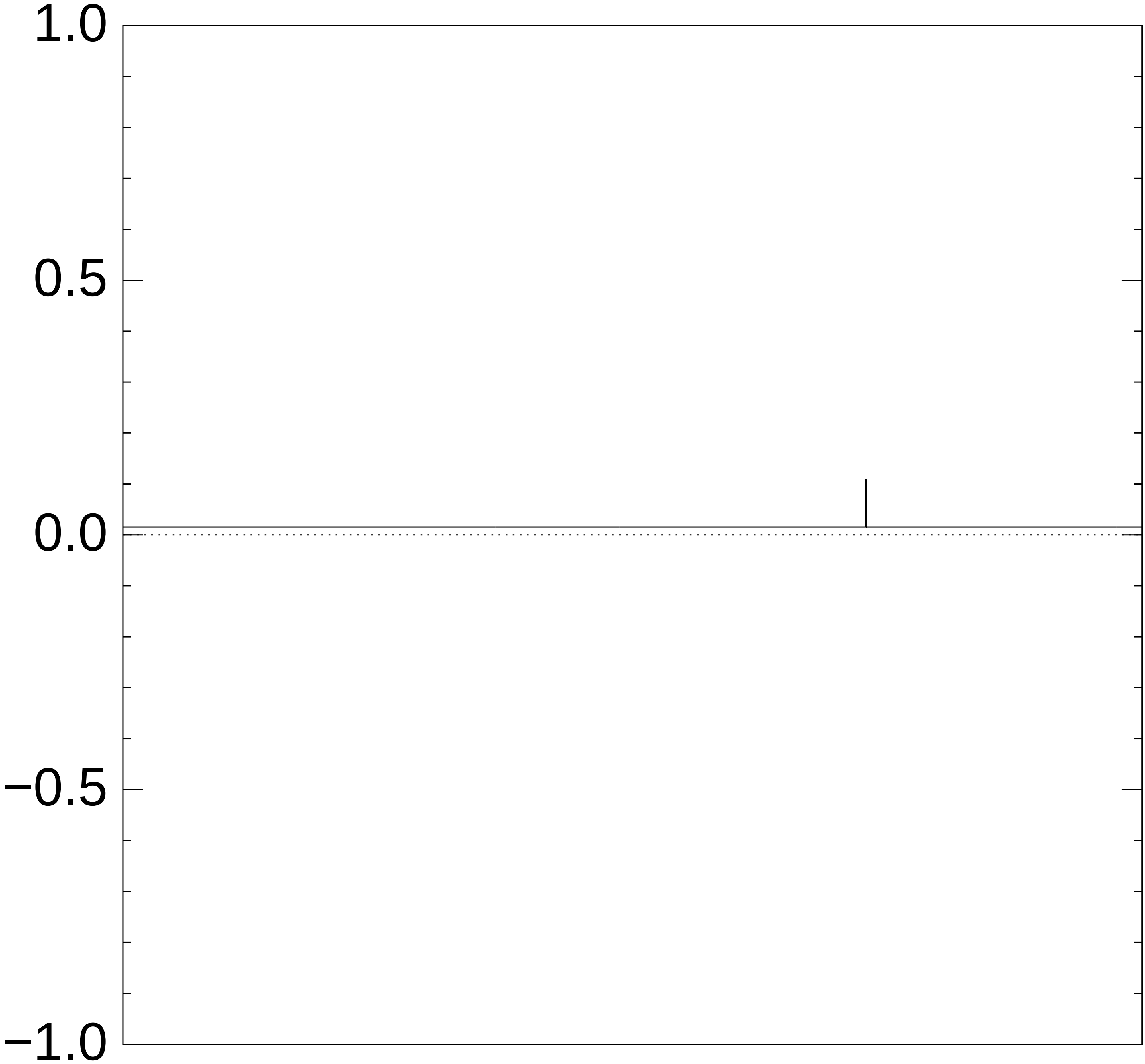
Normalized graph of $u \mapsto a_u$ for an example with $n = 12$ after Step $1 +$ Step $2 +$ Step 1:



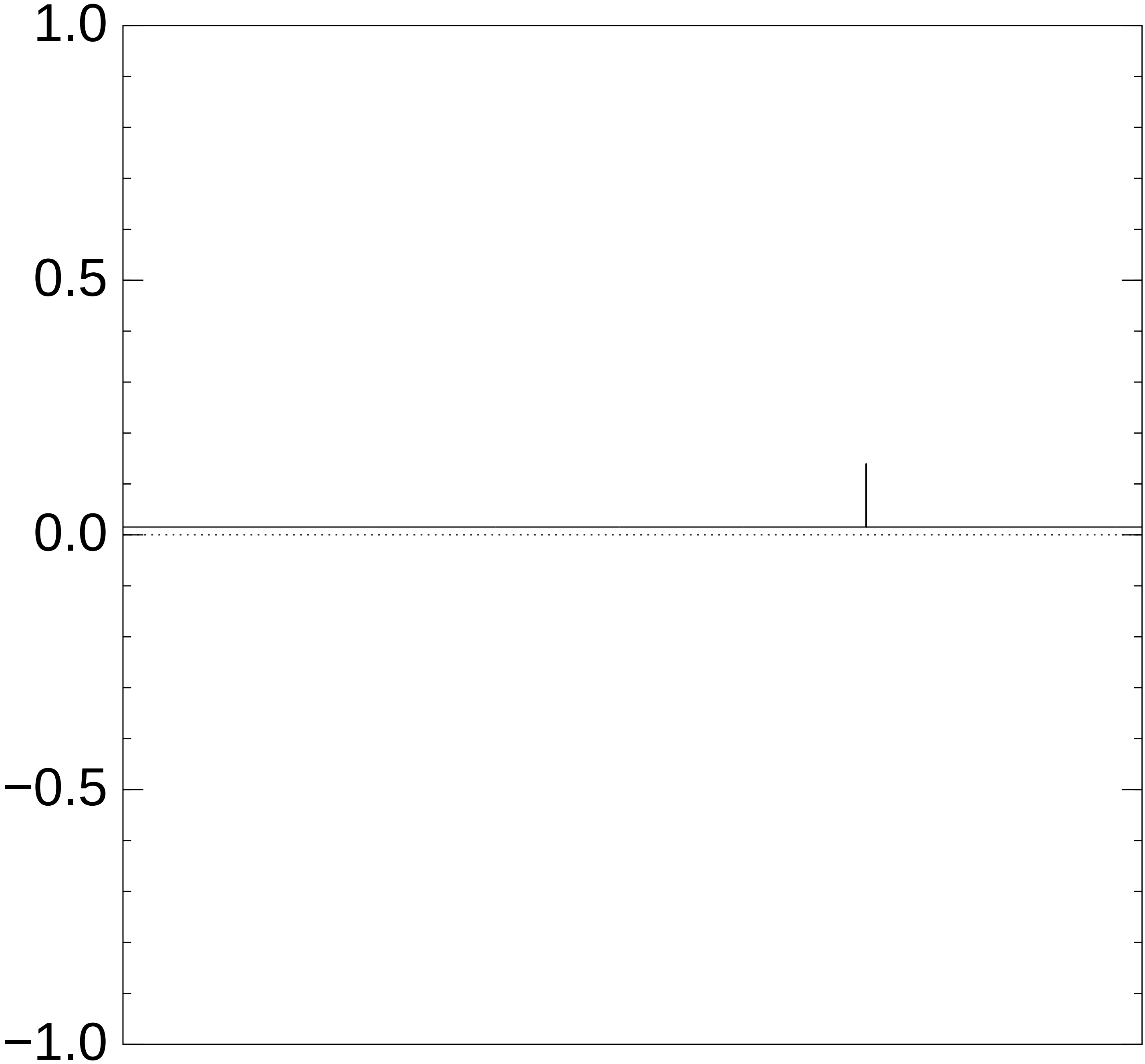
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $2 \times$ (Step 1 + Step 2):



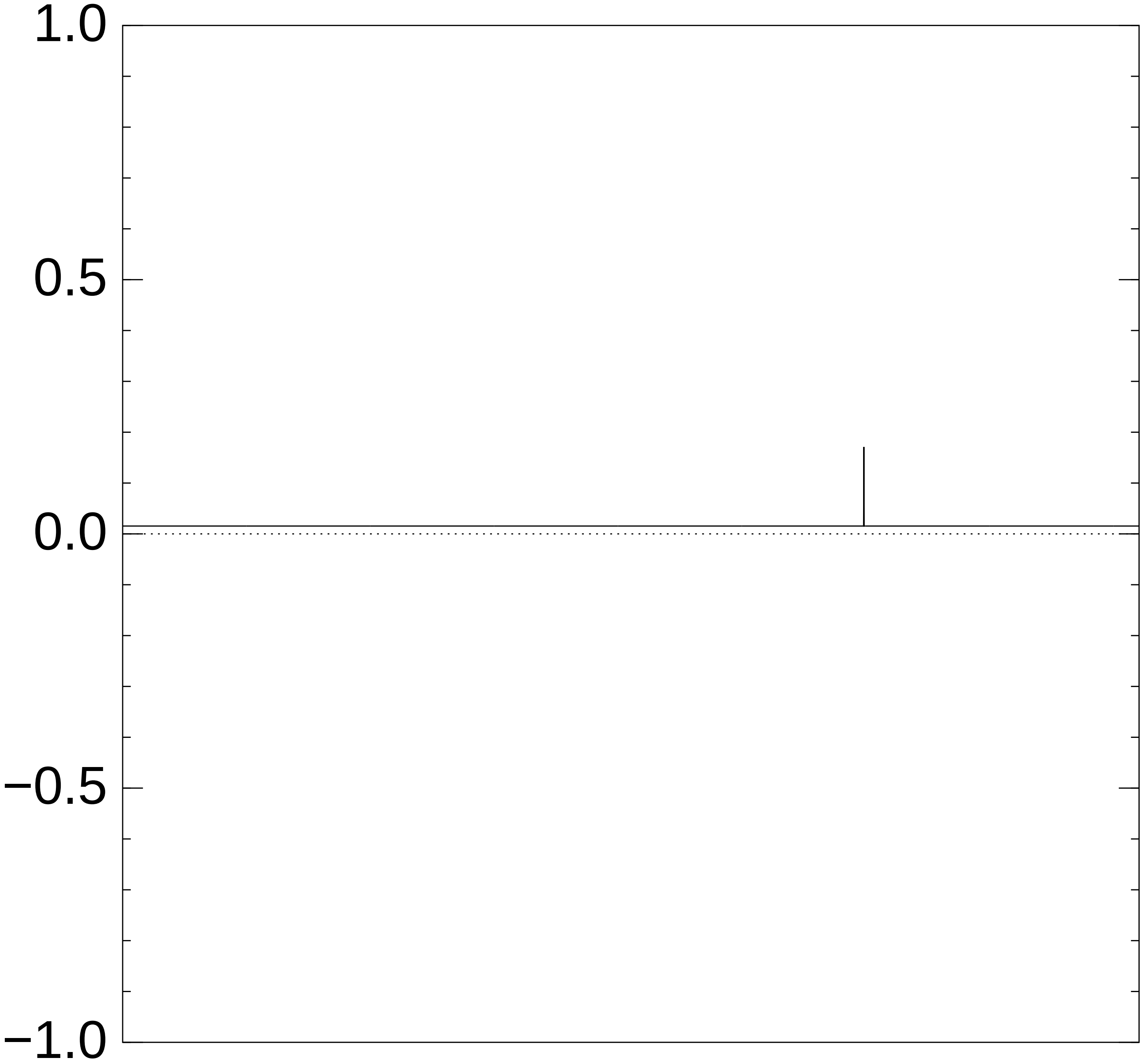
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $3 \times$ (Step 1 + Step 2):



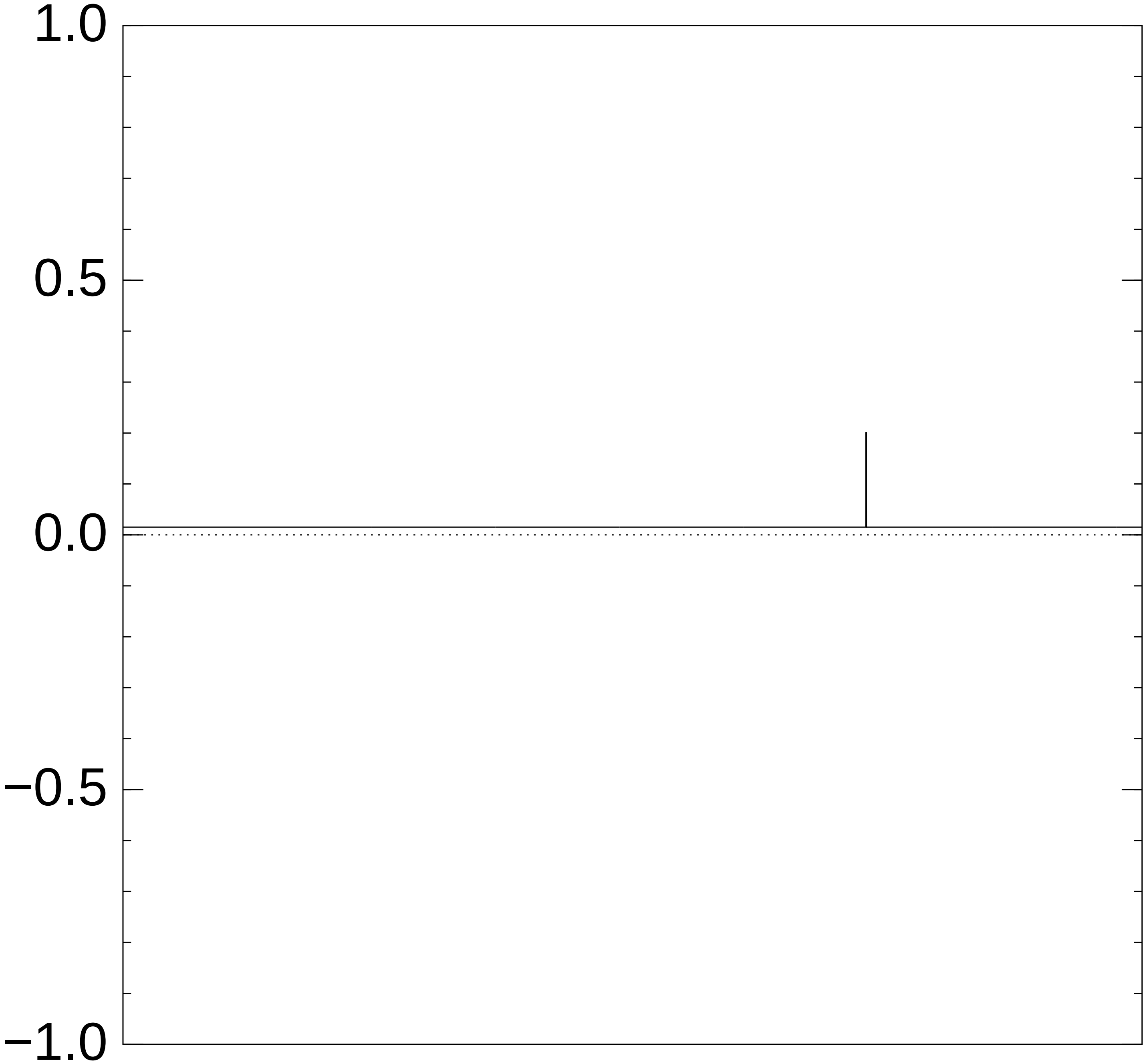
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $4 \times$ (Step $1 +$ Step 2):



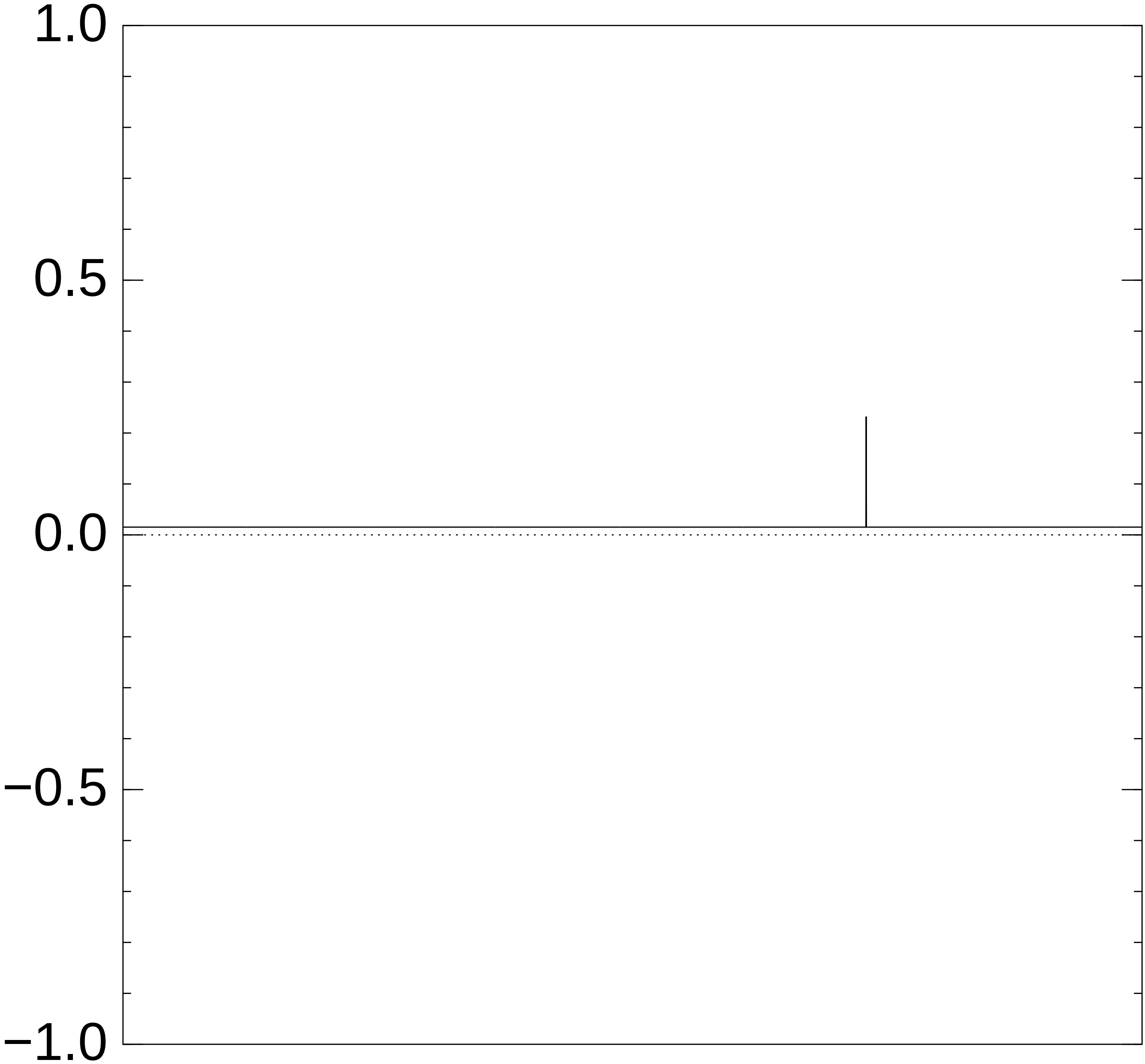
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $5 \times$ (Step 1 + Step 2):



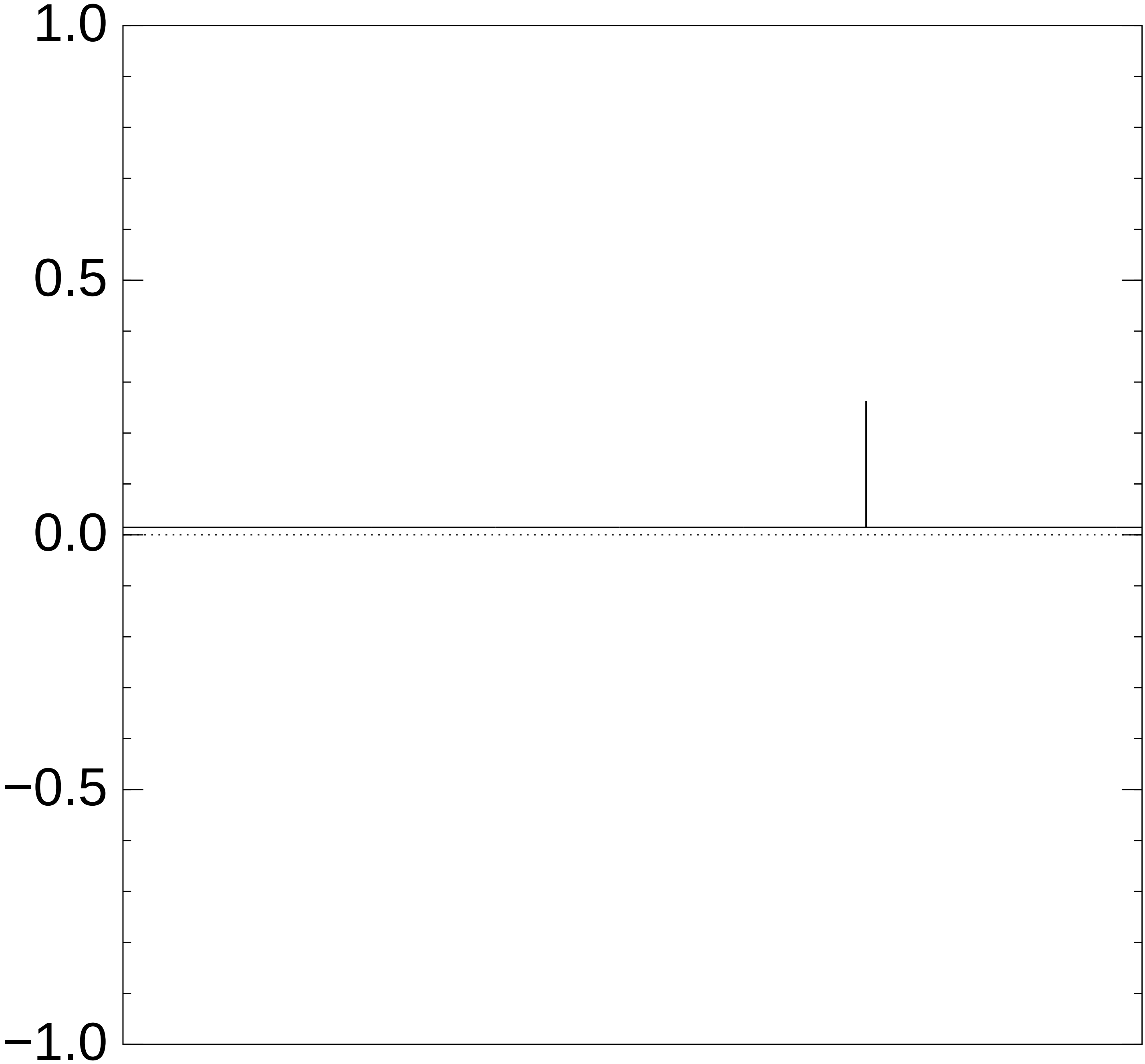
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $6 \times (\text{Step } 1 + \text{Step } 2)$:



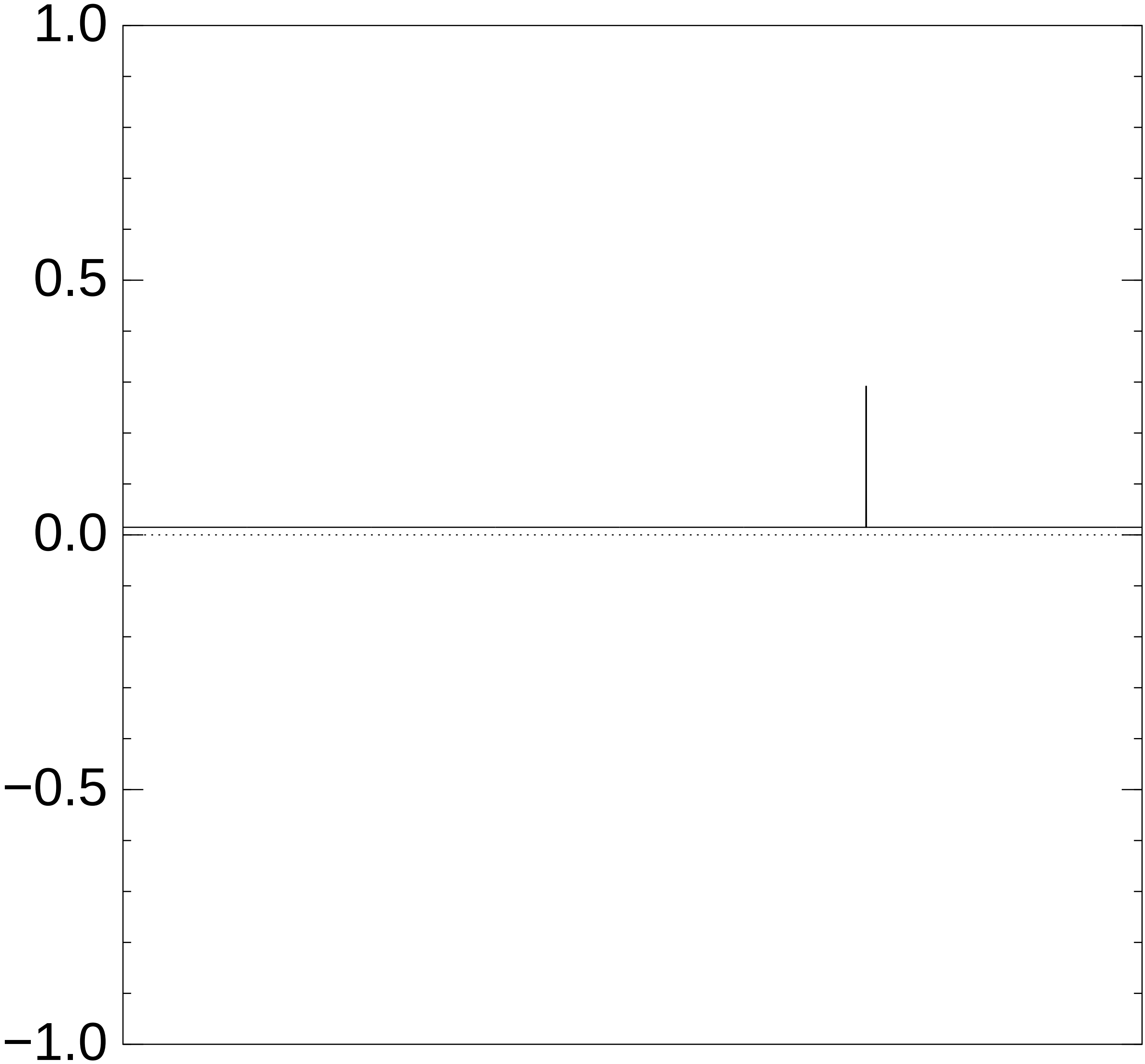
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $7 \times$ (Step 1 + Step 2):



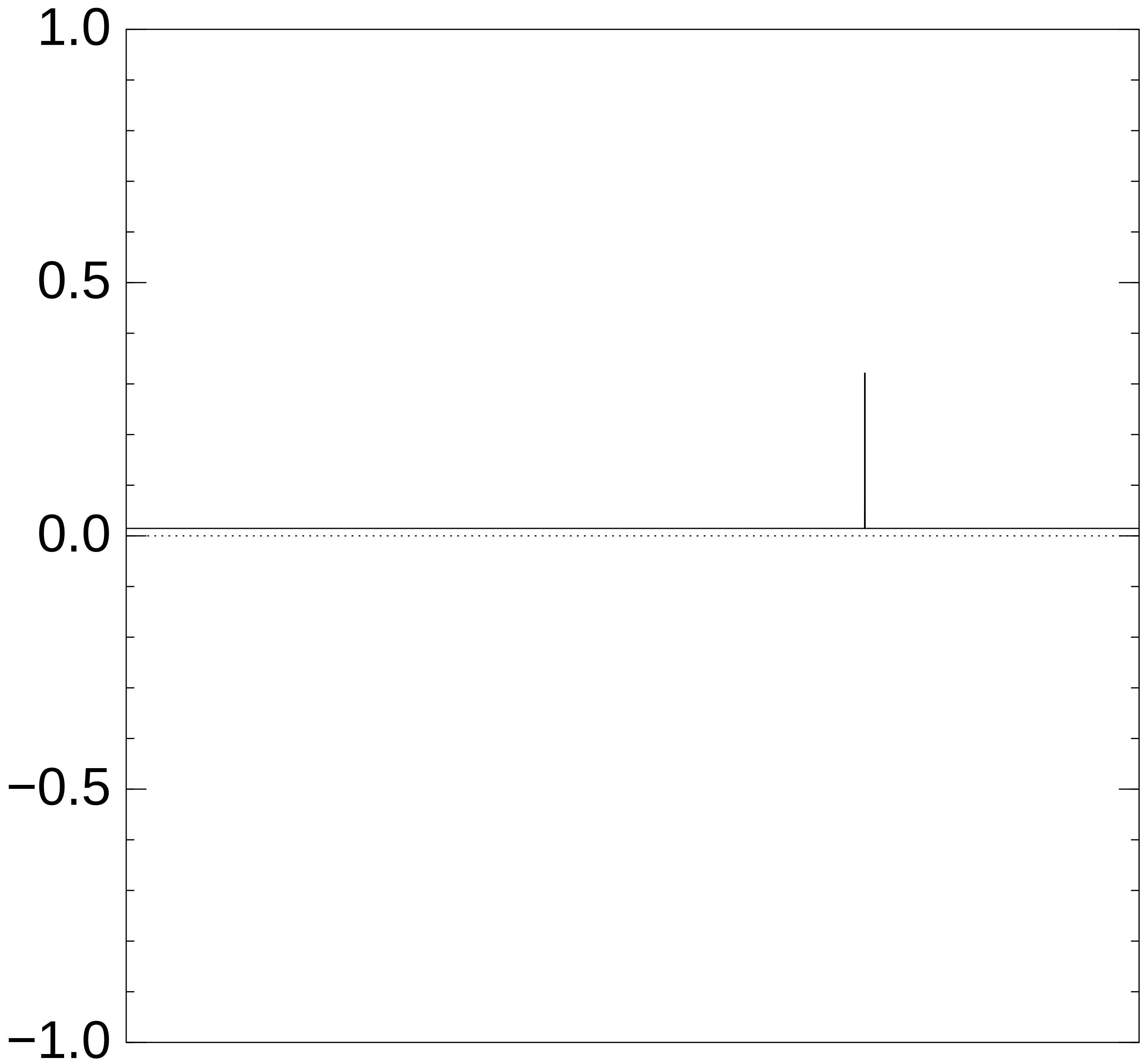
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $8 \times$ (Step 1 + Step 2):



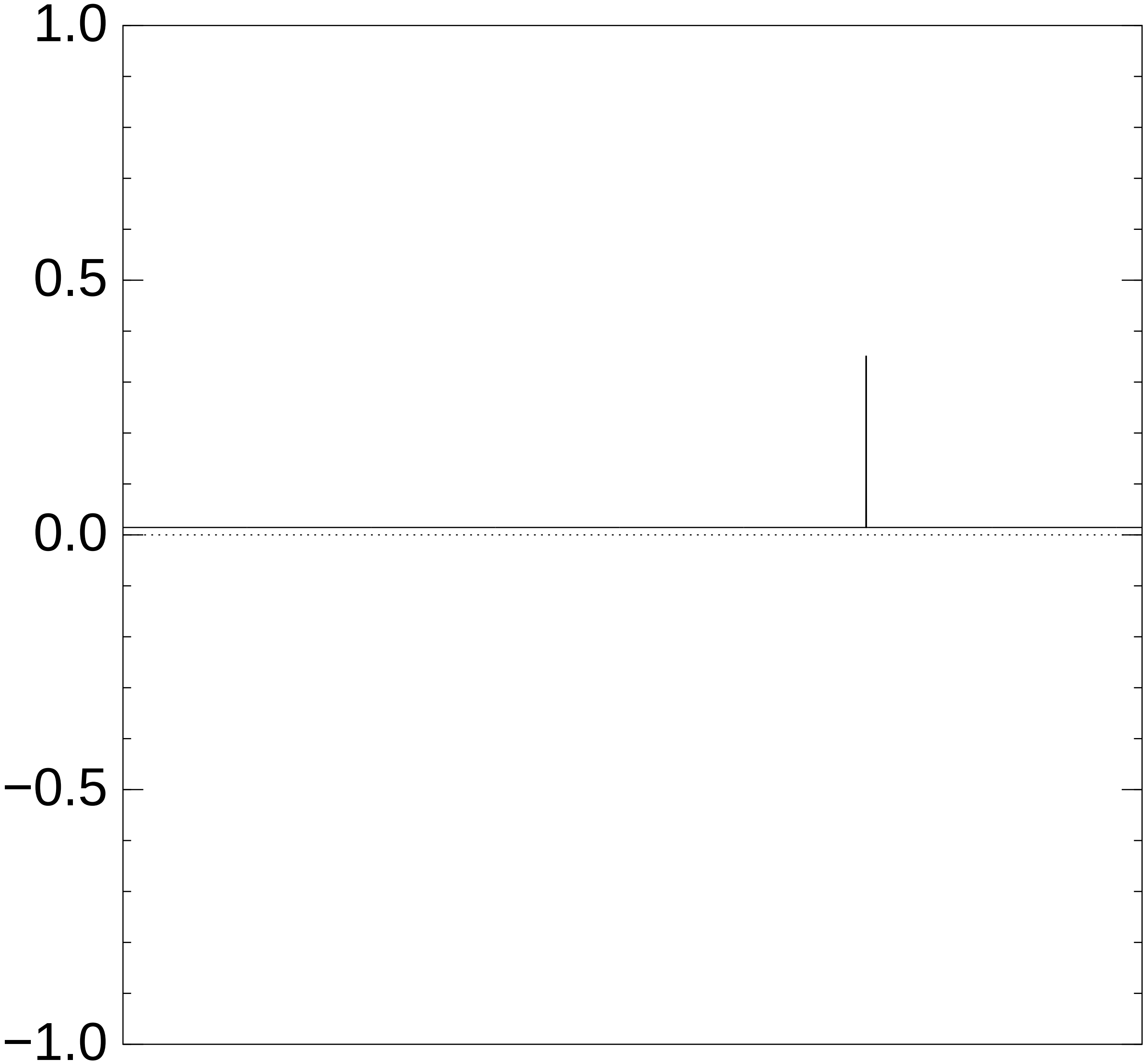
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $9 \times (\text{Step } 1 + \text{Step } 2)$:



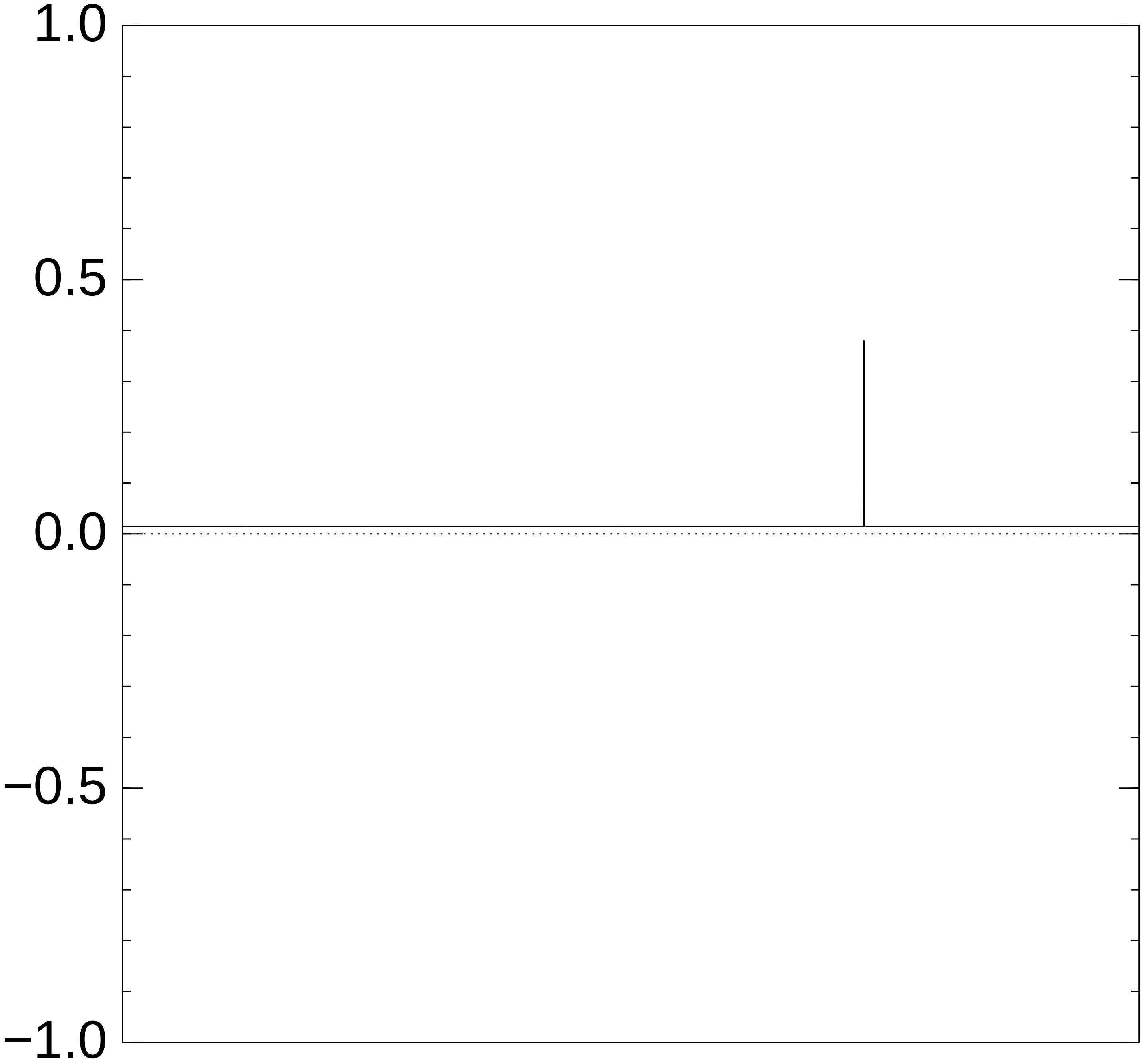
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $10 \times$ (Step $1 +$ Step 2):



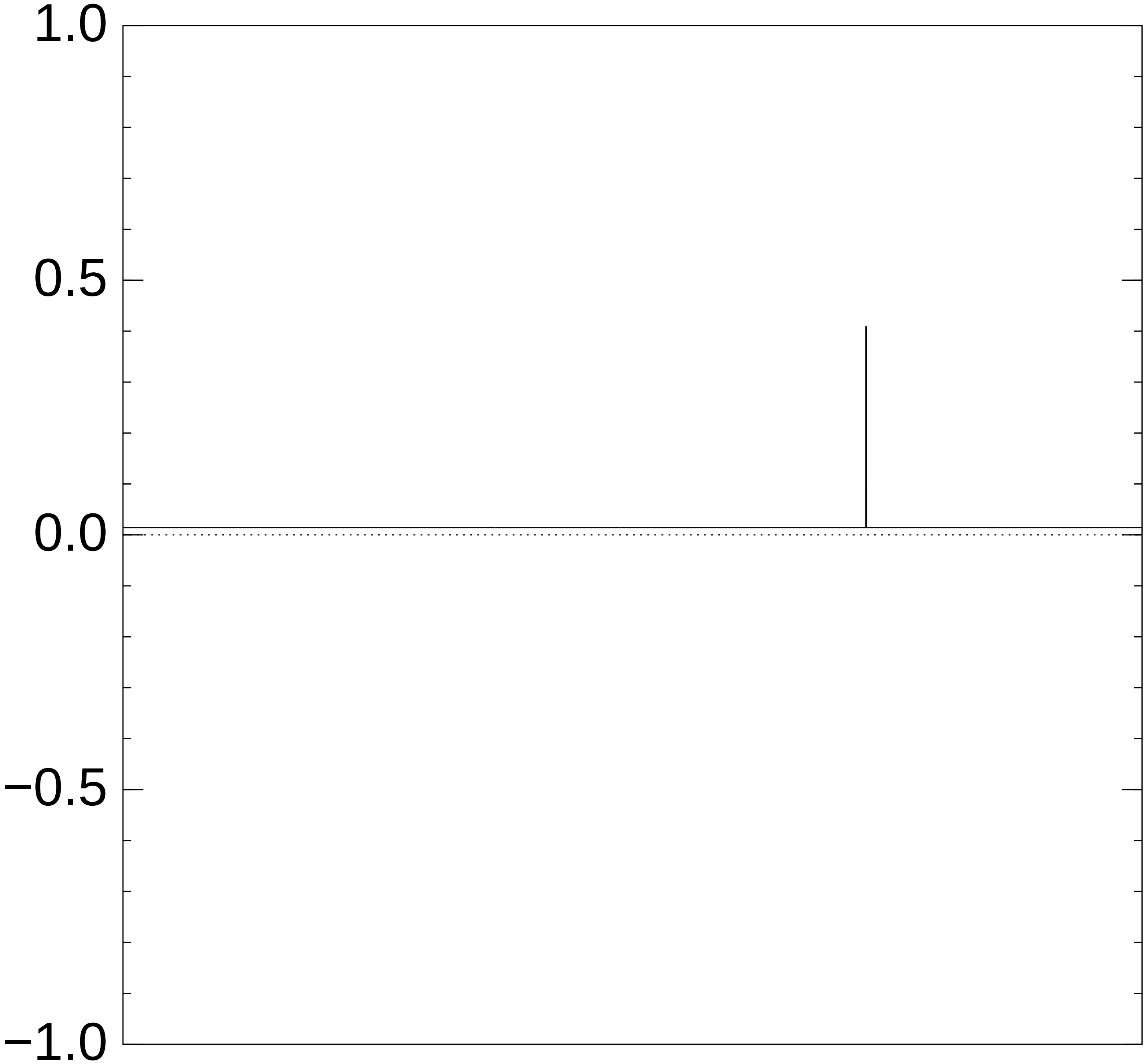
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $11 \times$ (Step 1 + Step 2):



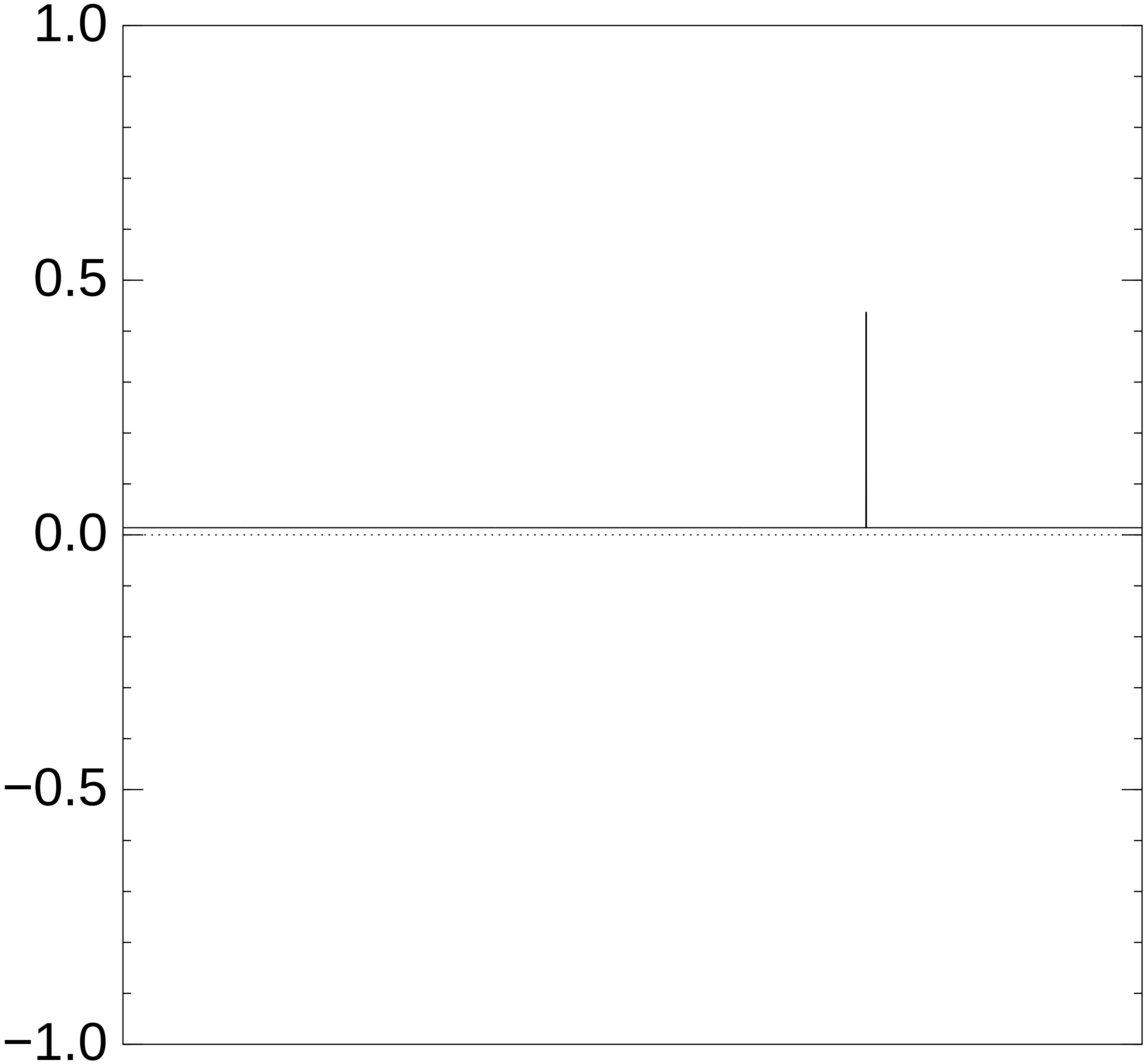
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $12 \times$ (Step 1 + Step 2):



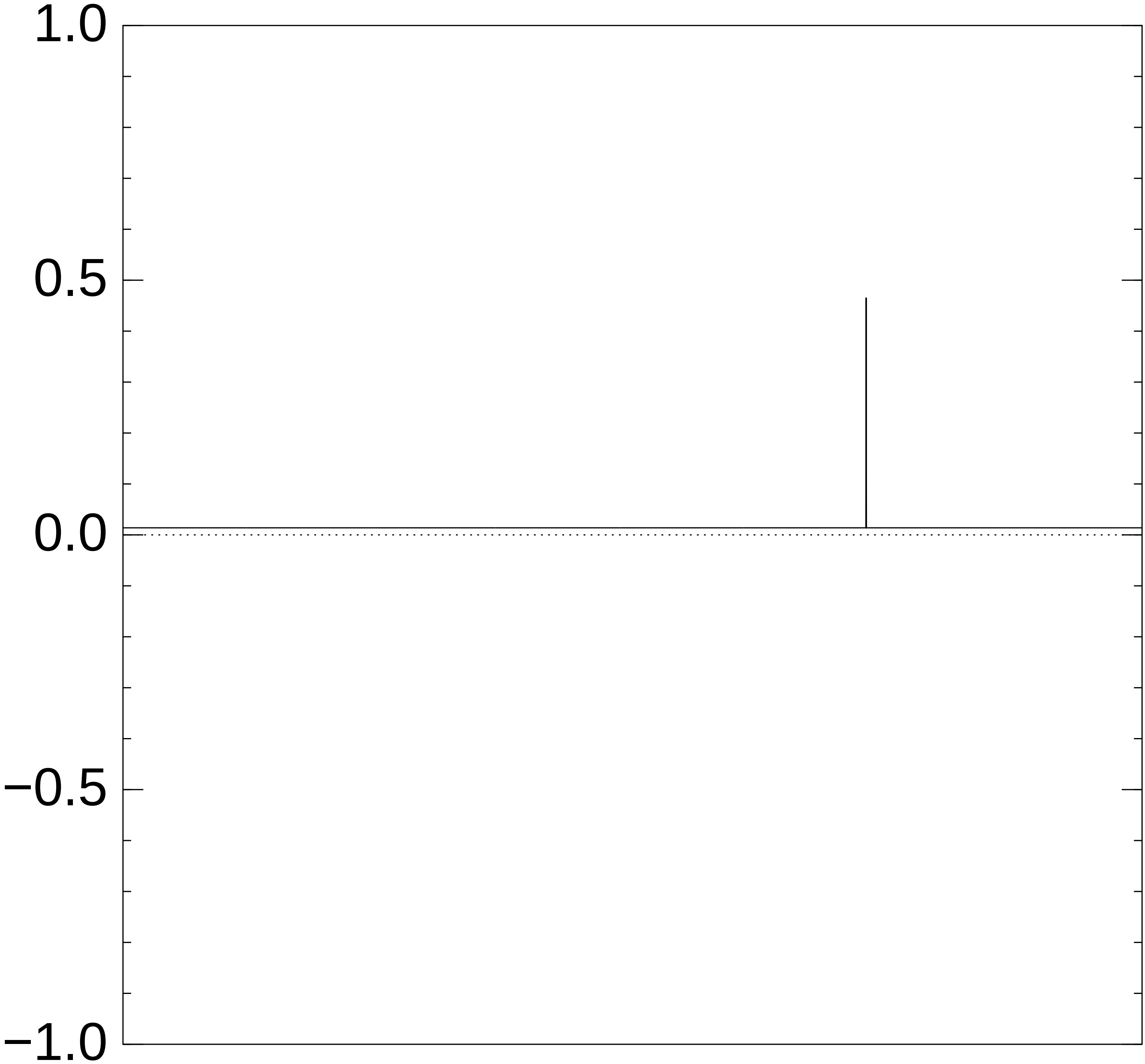
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $13 \times (\text{Step } 1 + \text{Step } 2)$:



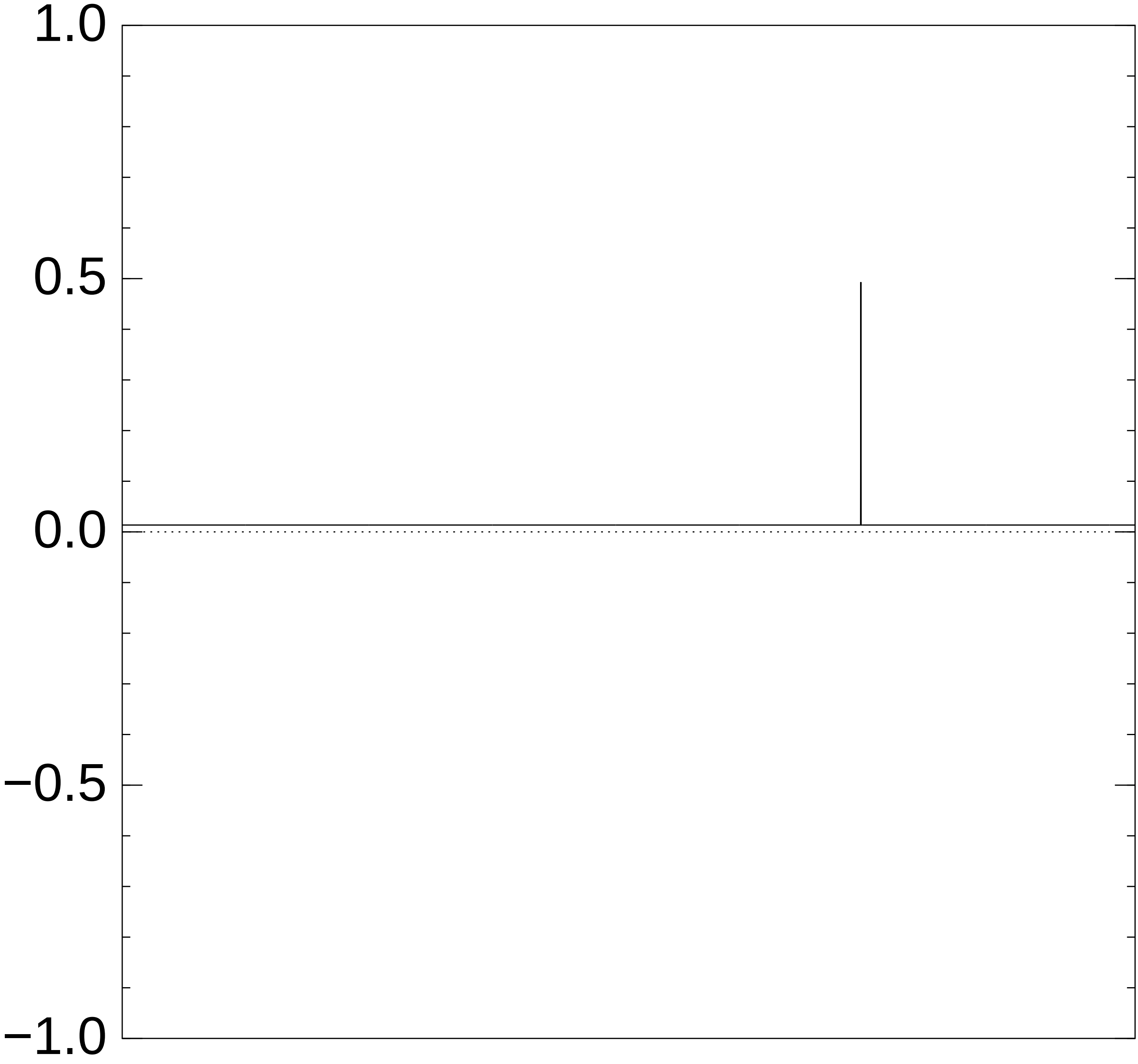
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $14 \times (\text{Step } 1 + \text{Step } 2)$:



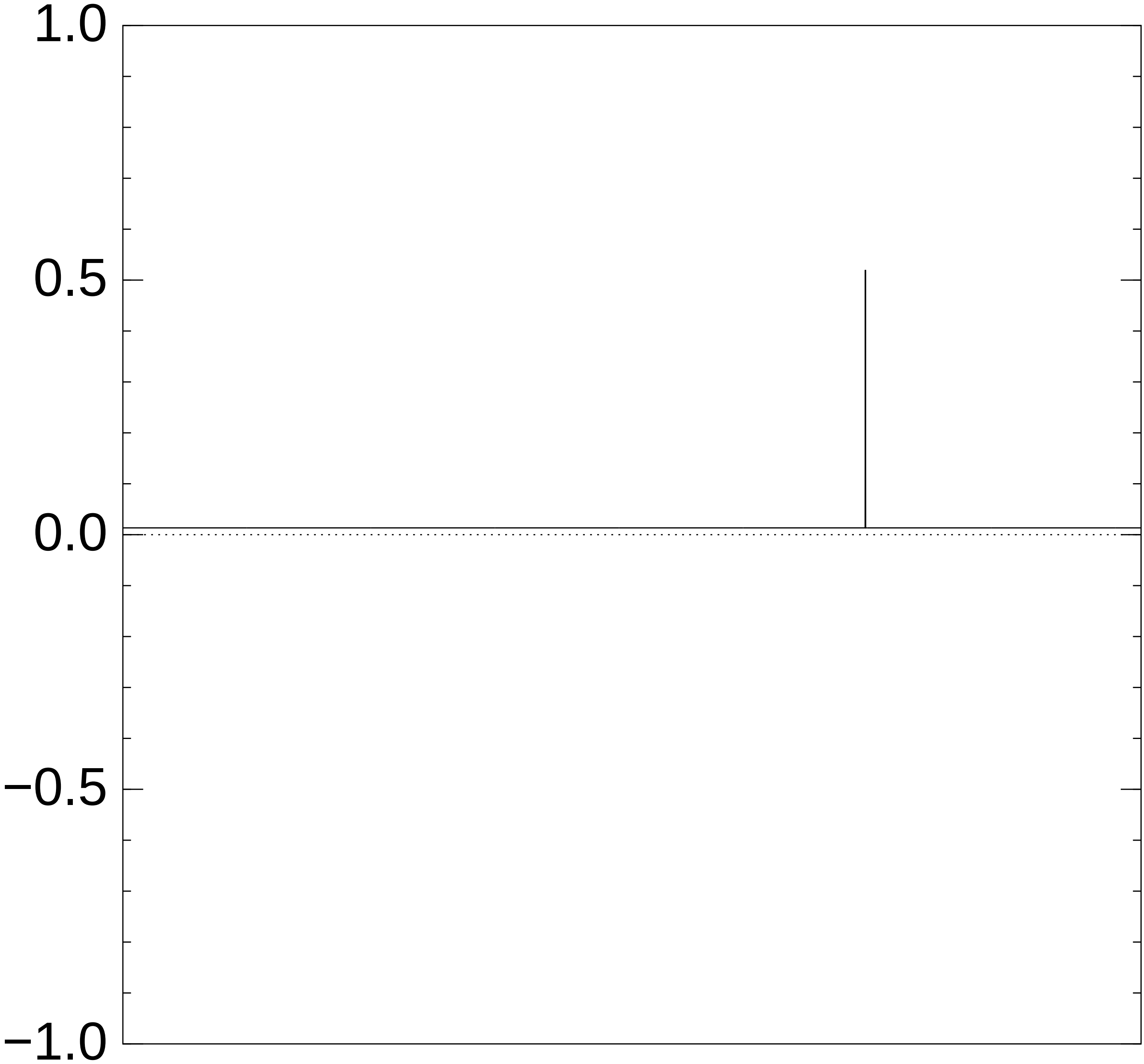
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $15 \times (\text{Step } 1 + \text{Step } 2)$:



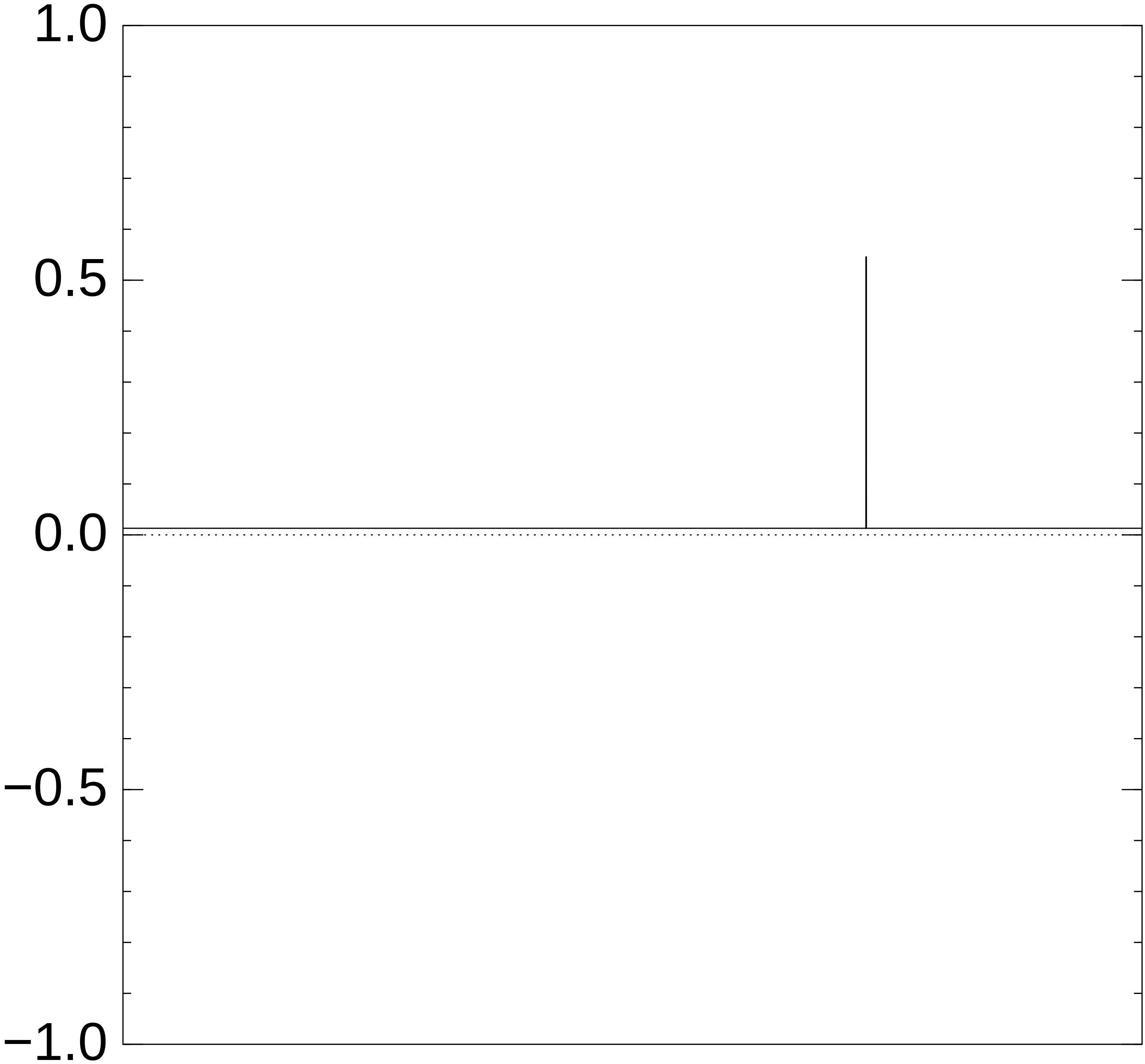
Normalized graph of $u \mapsto a_u$ for an example with $n = 12$ after $16 \times (\text{Step 1} + \text{Step 2})$:



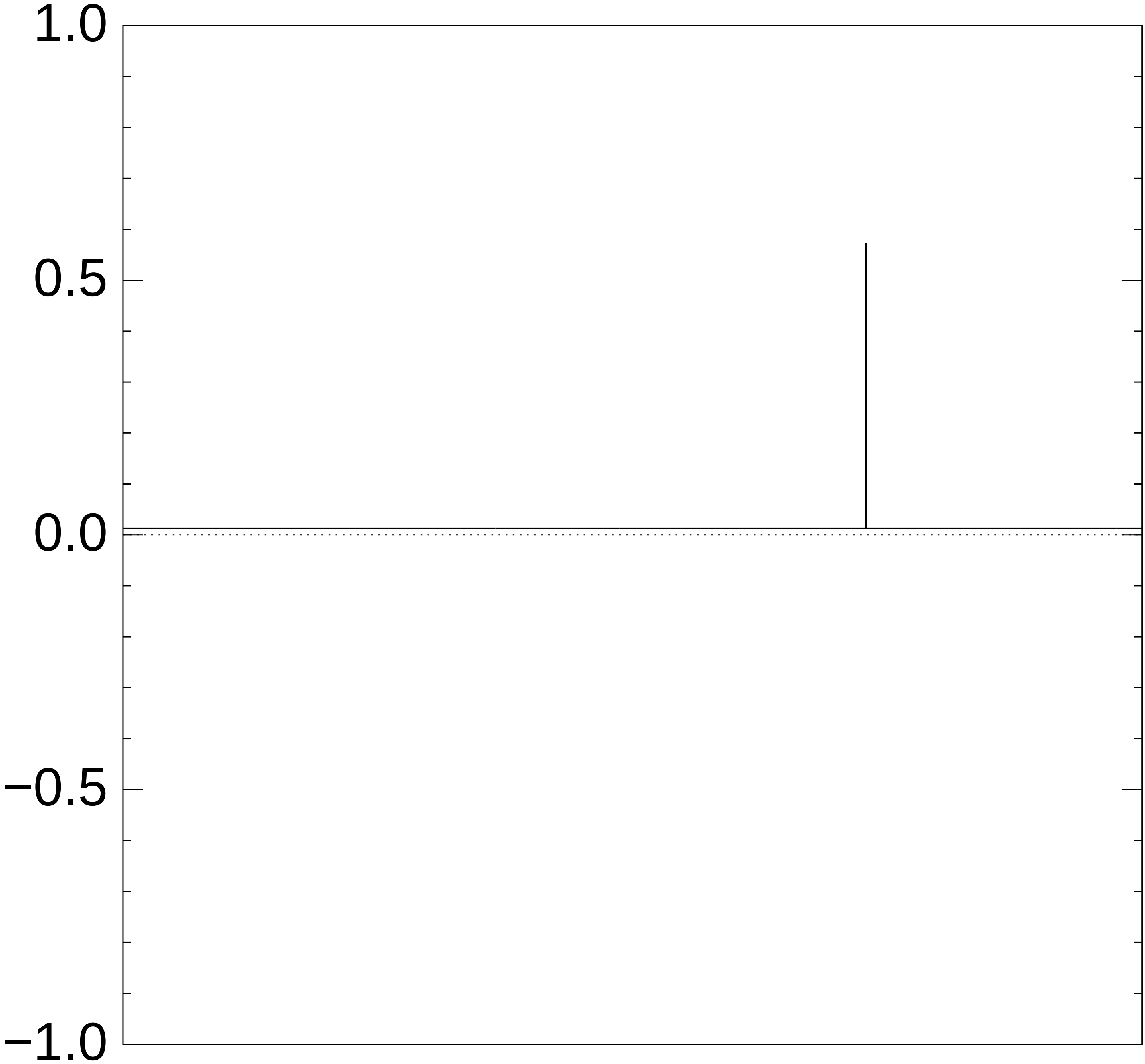
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $17 \times$ (Step $1 +$ Step 2):



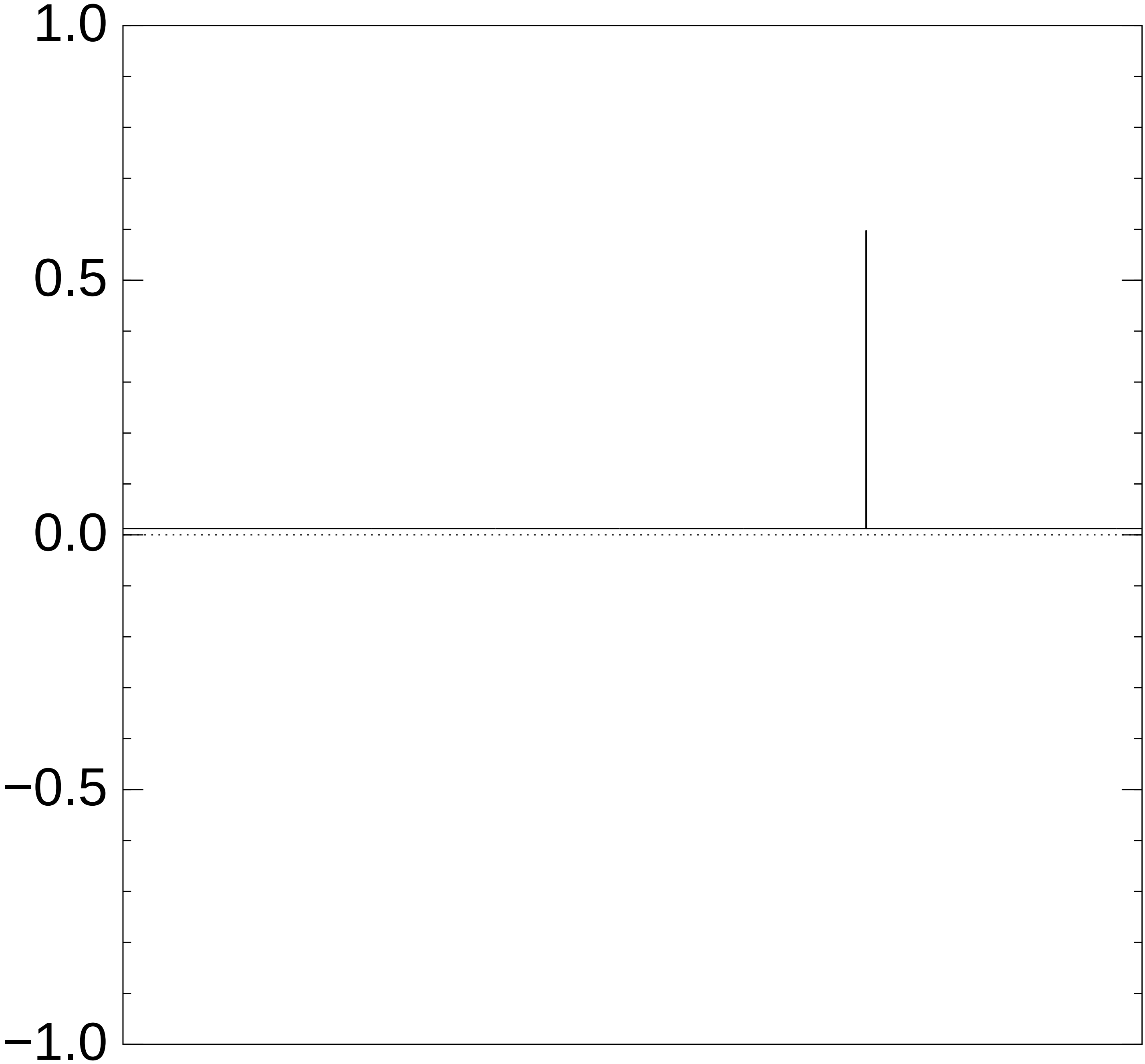
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $18 \times (\text{Step 1} + \text{Step 2})$:



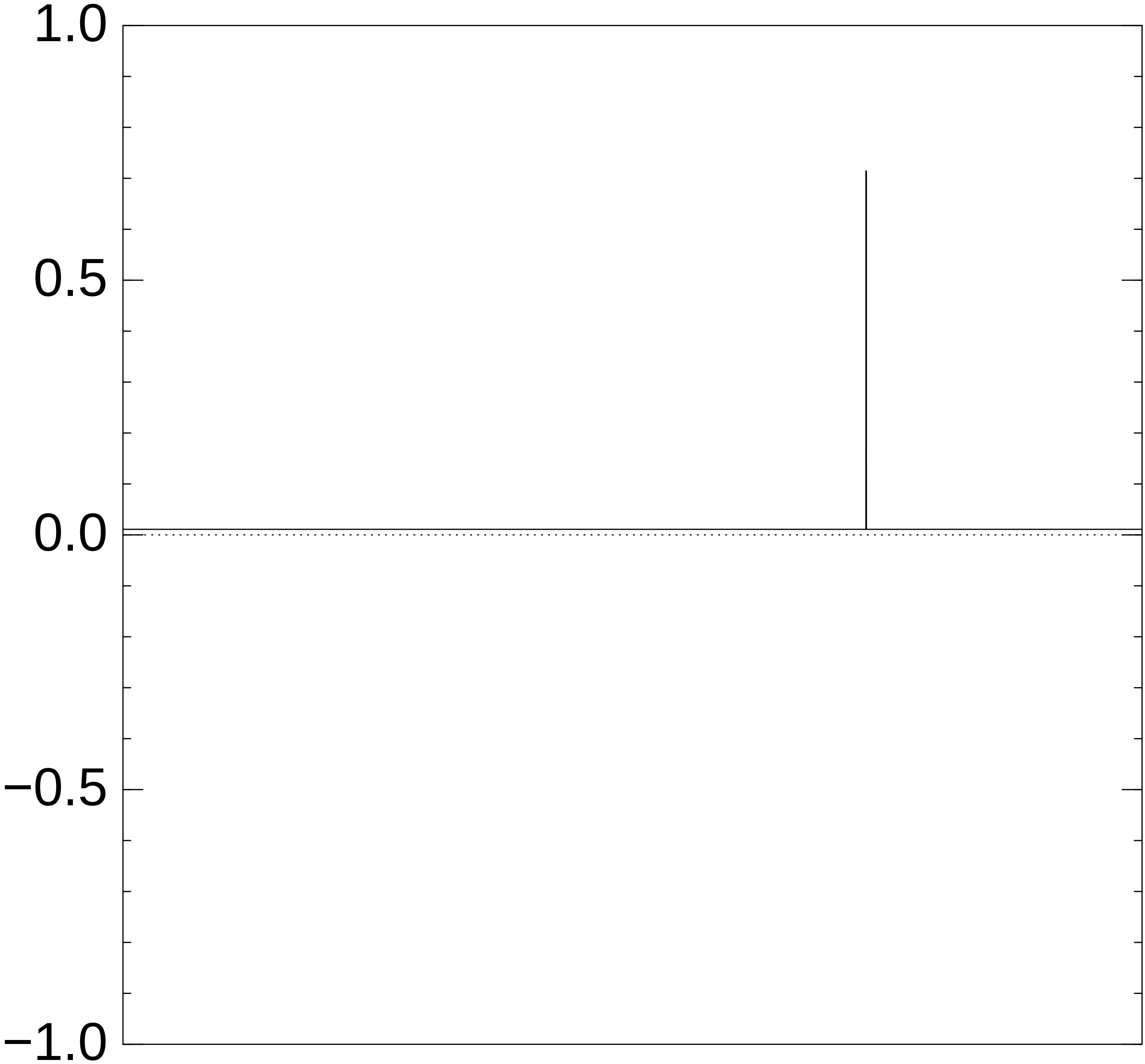
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $19 \times$ (Step $1 +$ Step 2):



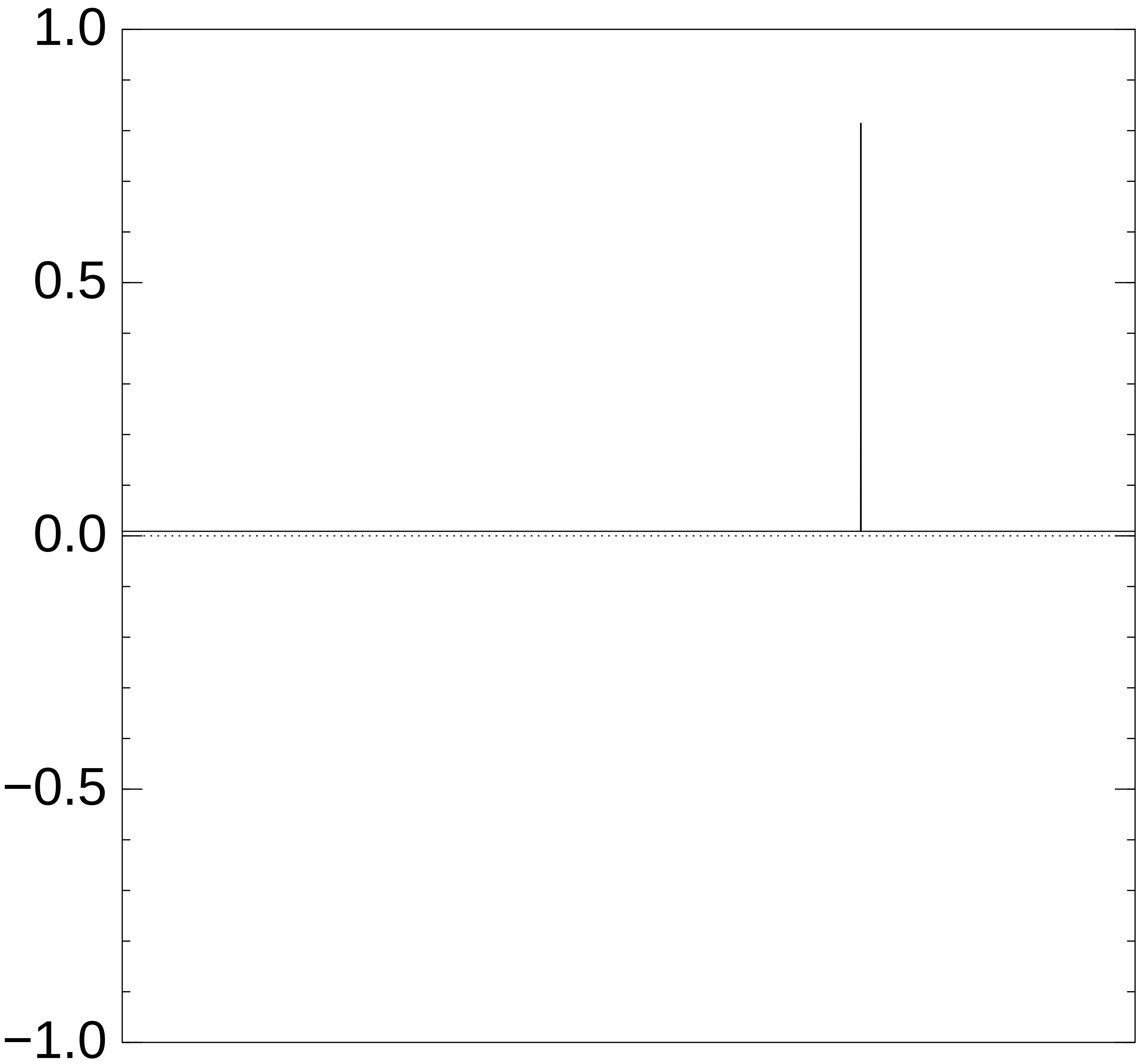
Normalized graph of $u \mapsto a_u$ for an example with $n = 12$ after $20 \times (\text{Step } 1 + \text{Step } 2)$:



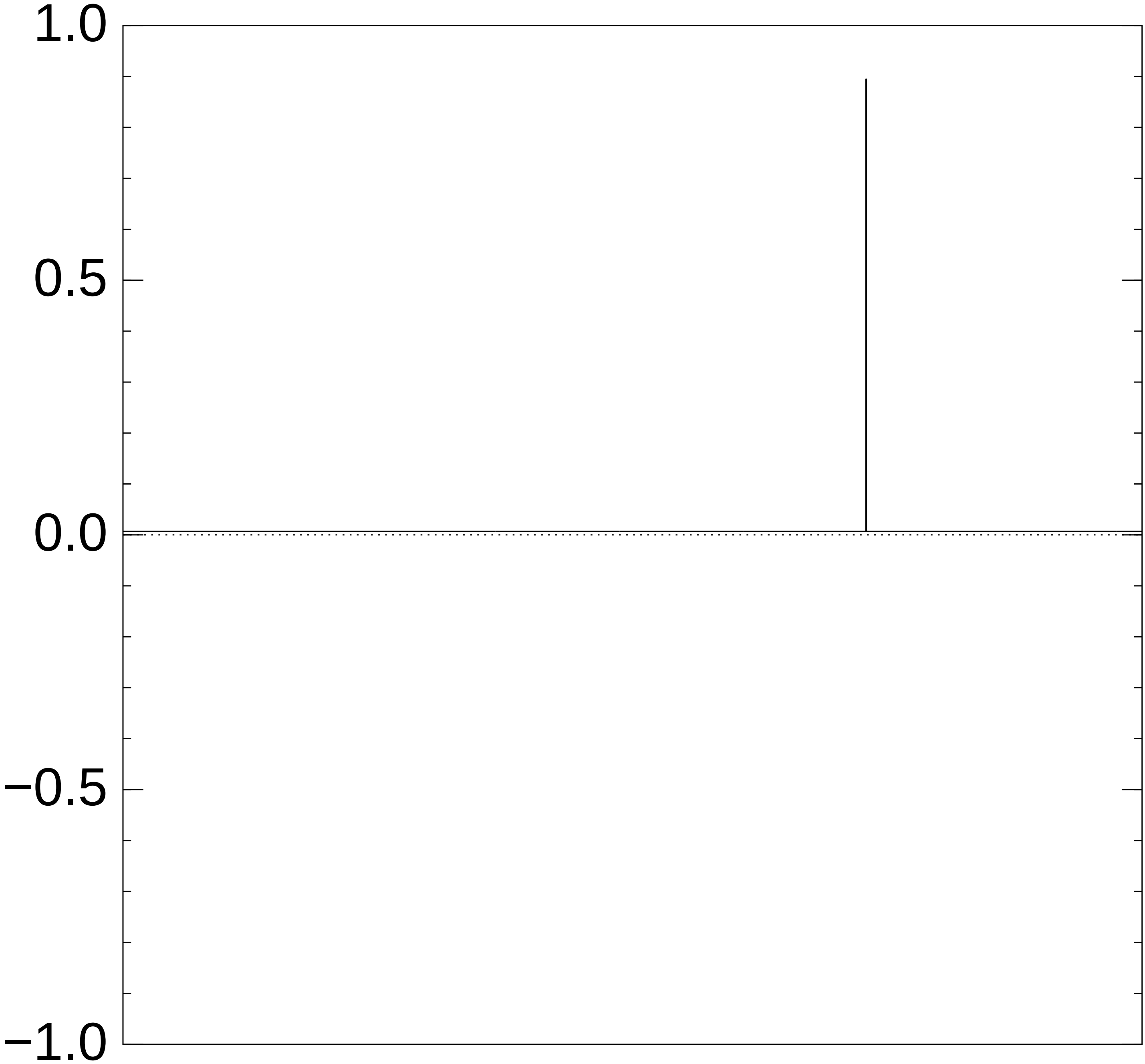
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $25 \times (\text{Step } 1 + \text{Step } 2)$:



Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $30 \times$ (Step 1 + Step 2):

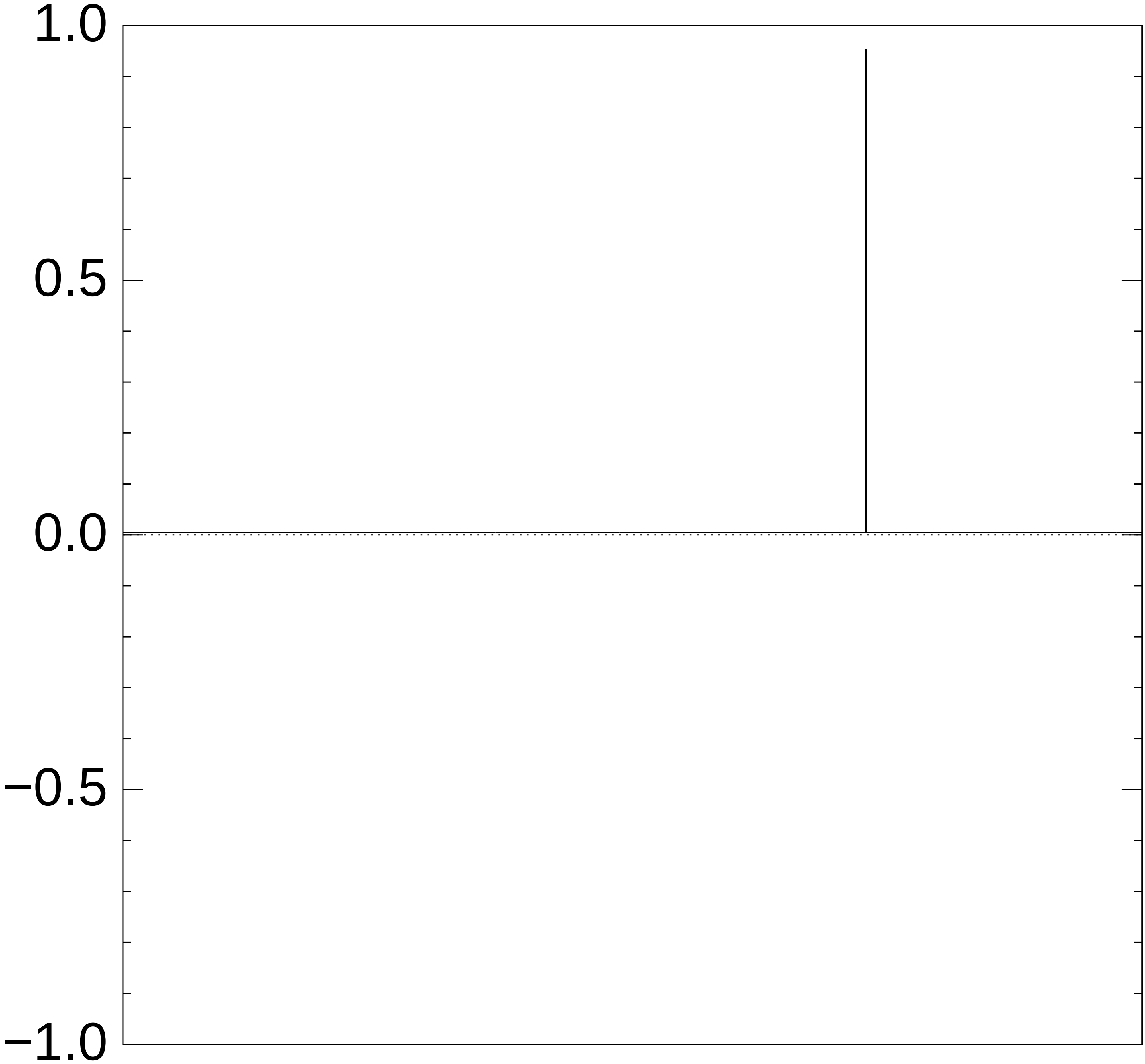


Normalized graph of $u \mapsto a_u$ for an example with $n = 12$ after $35 \times$ (Step 1 + Step 2):

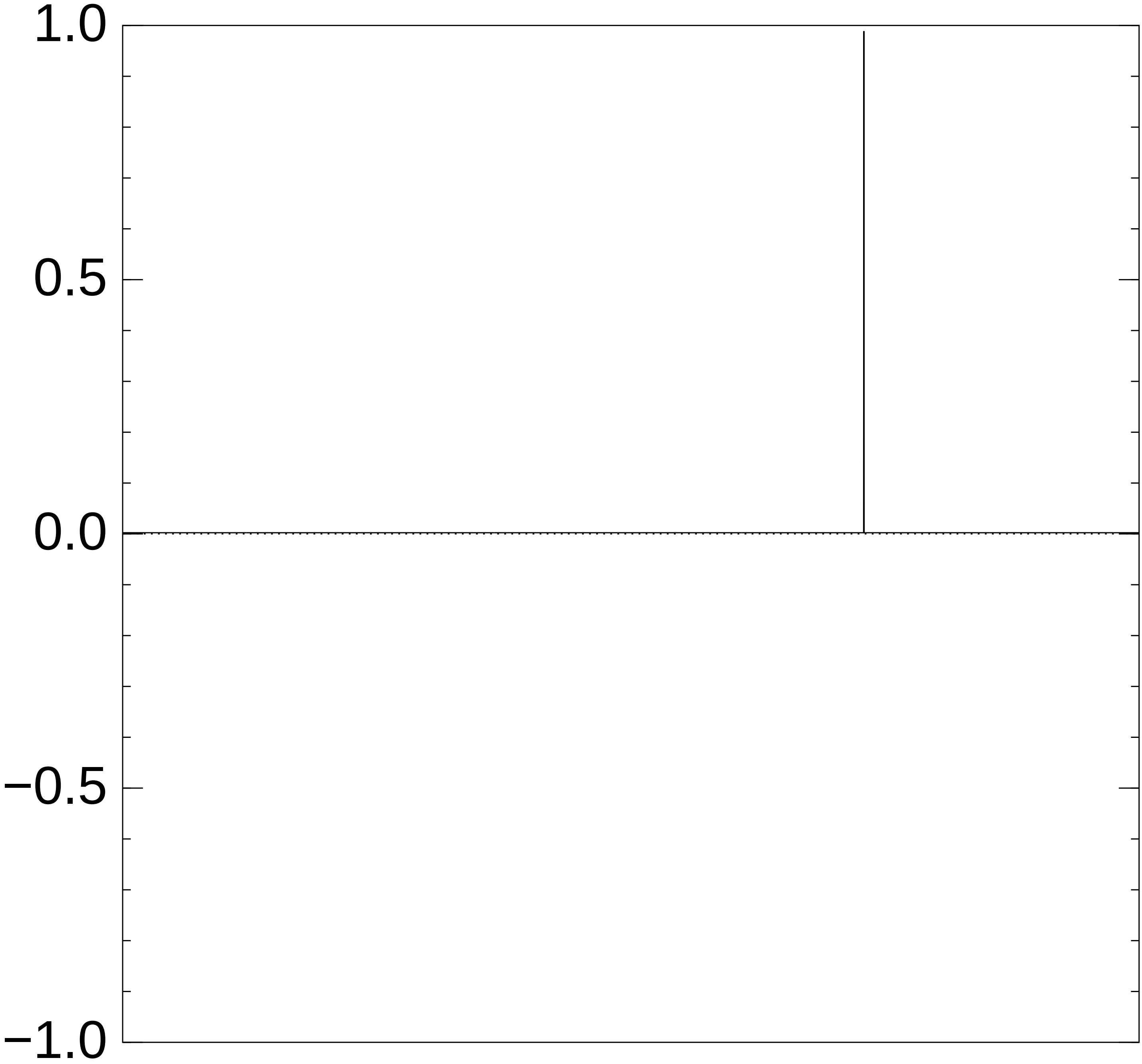


Good moment to stop, measure.

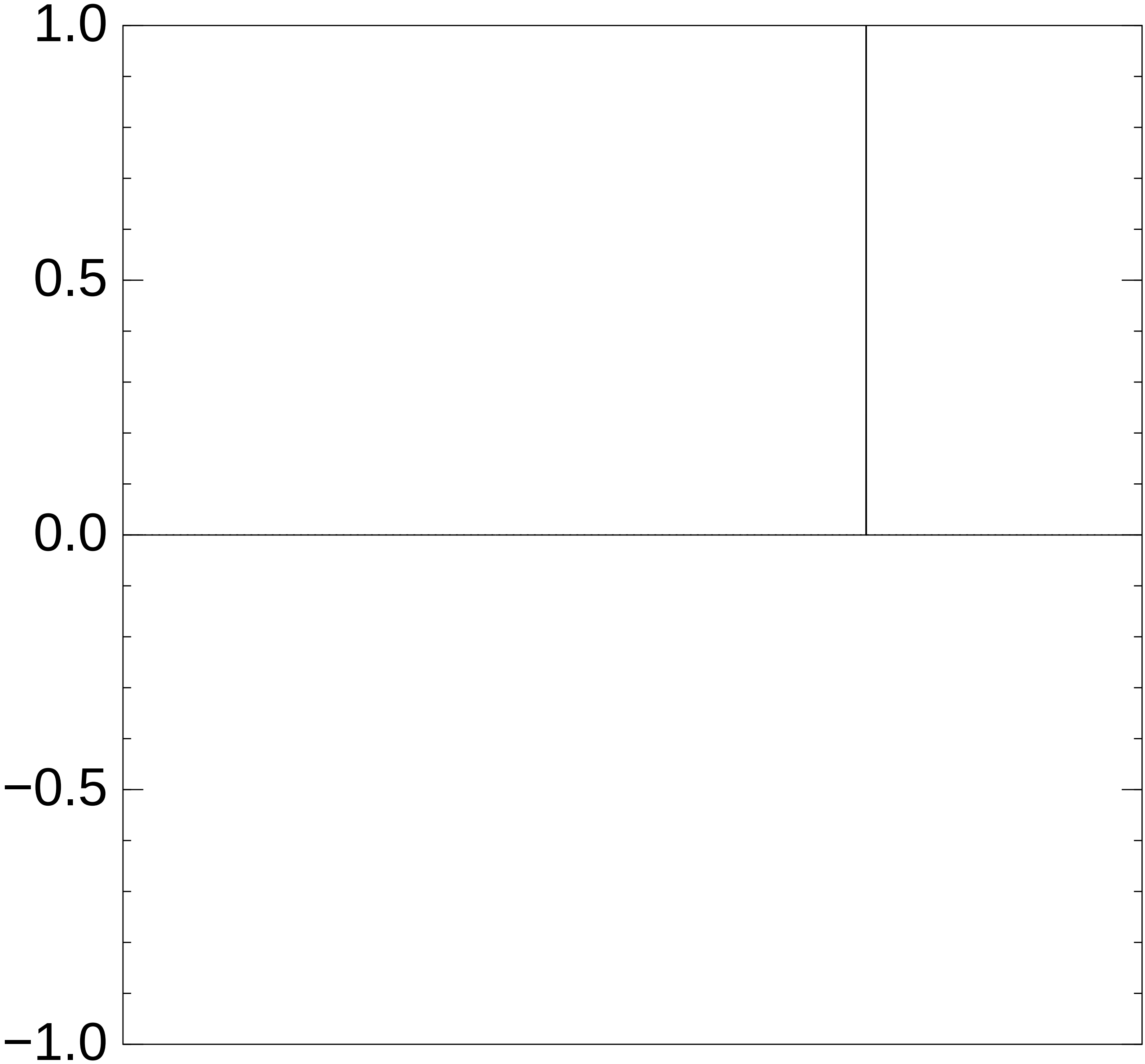
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $40 \times$ (Step $1 +$ Step 2):



Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $45 \times (\text{Step } 1 + \text{Step } 2)$:

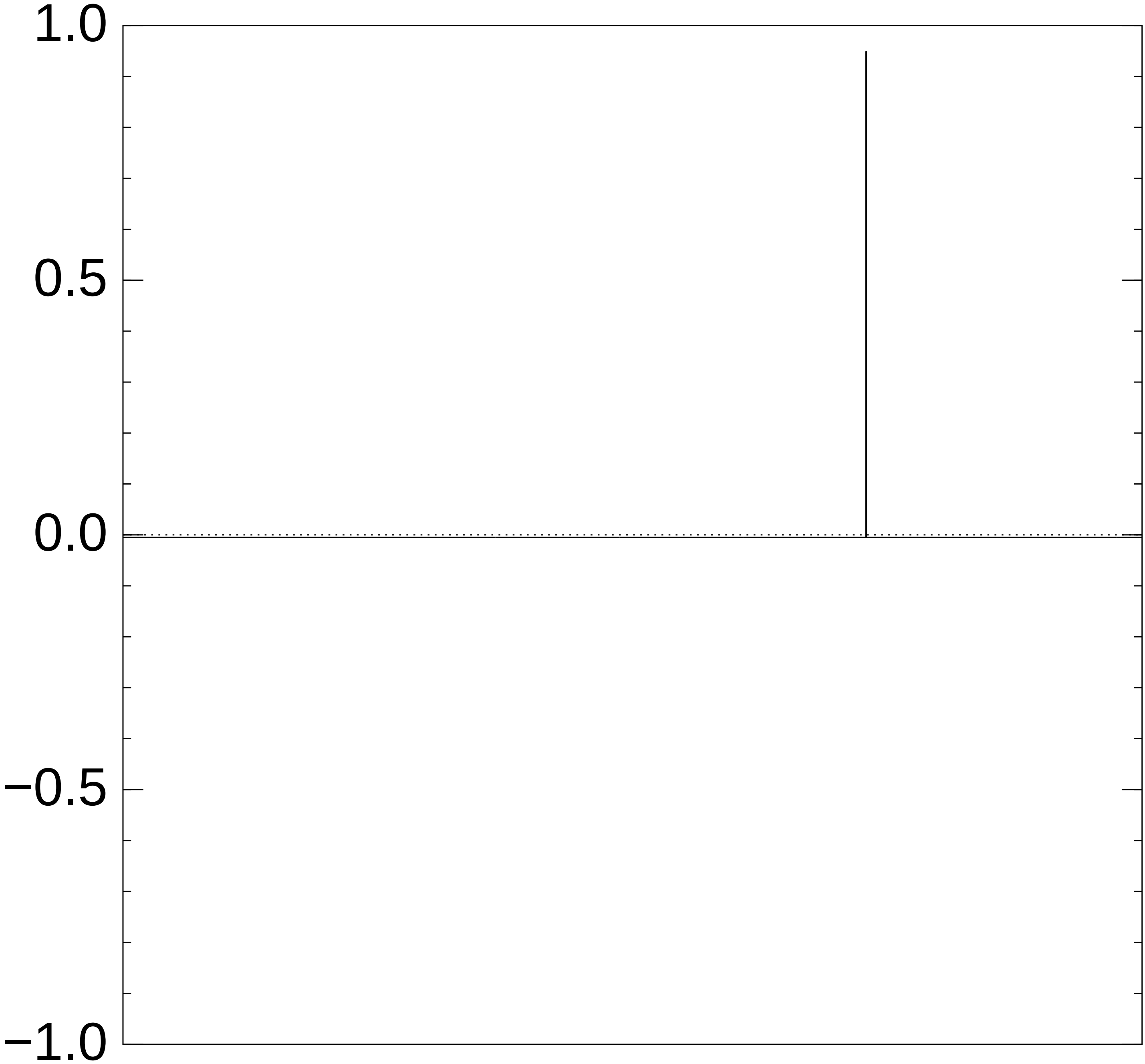


Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $50 \times$ (Step $1 +$ Step 2):

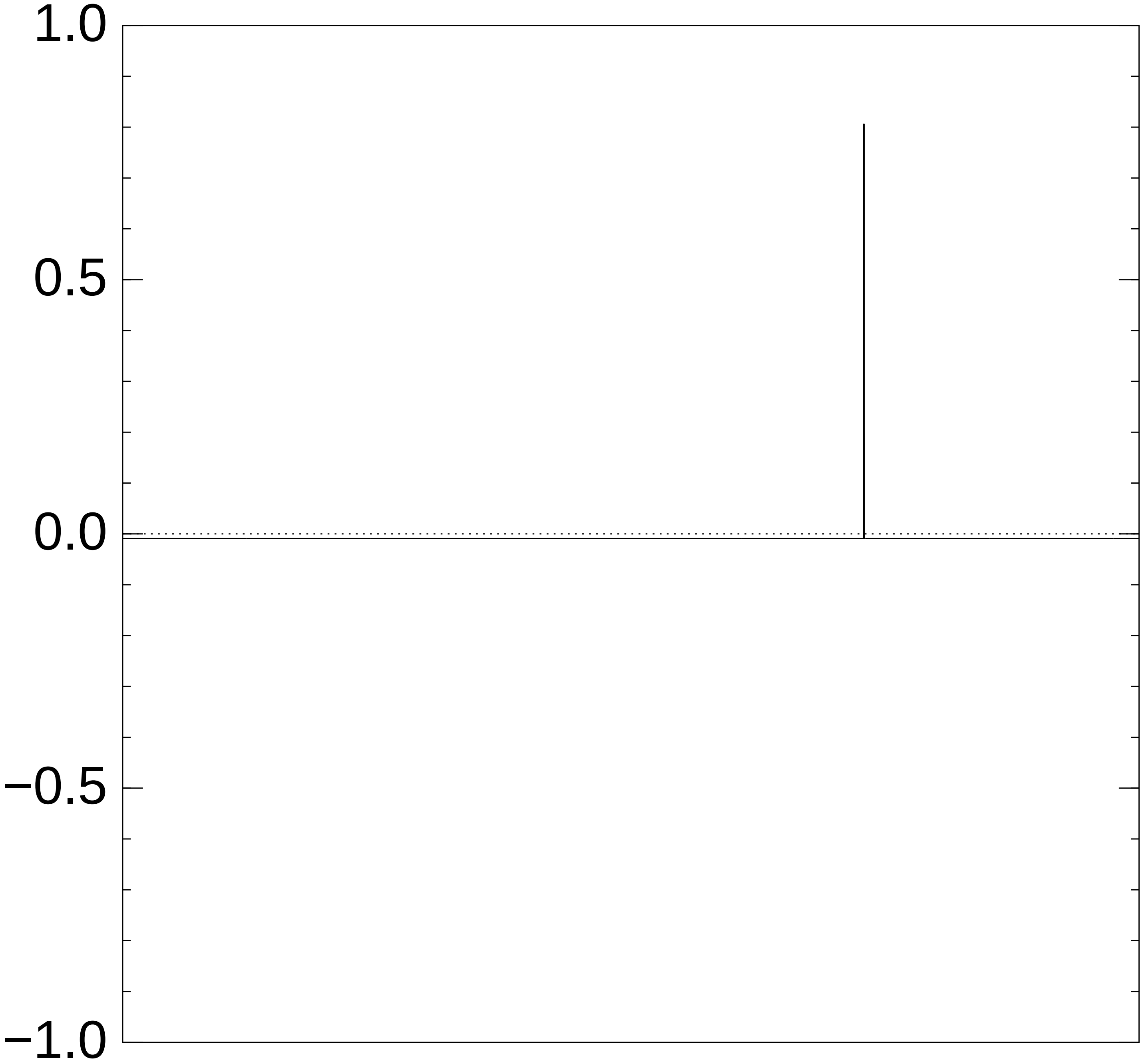


Traditional stopping point.

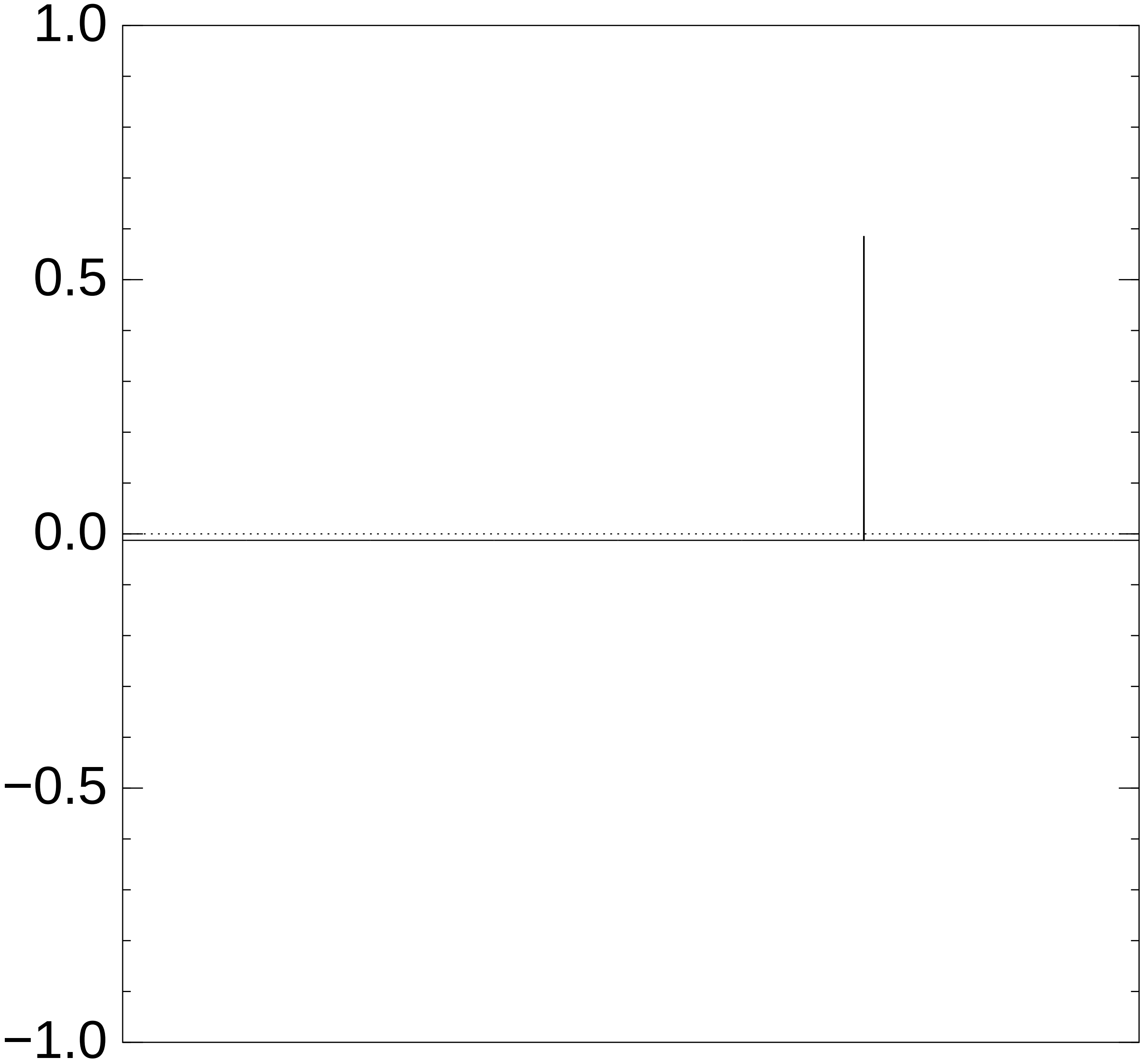
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $60 \times (\text{Step } 1 + \text{Step } 2)$:



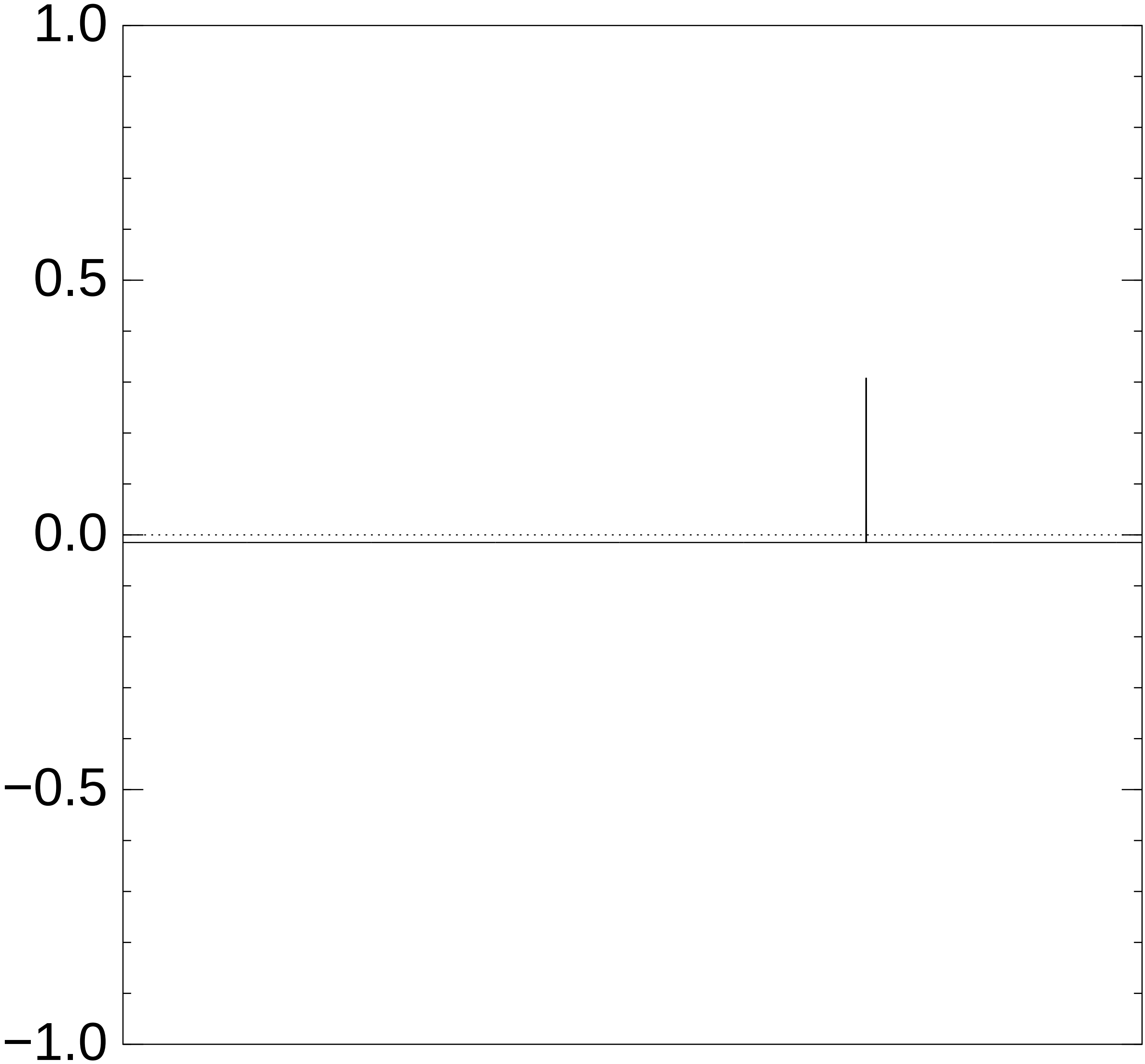
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $70 \times$ (Step $1 +$ Step 2):



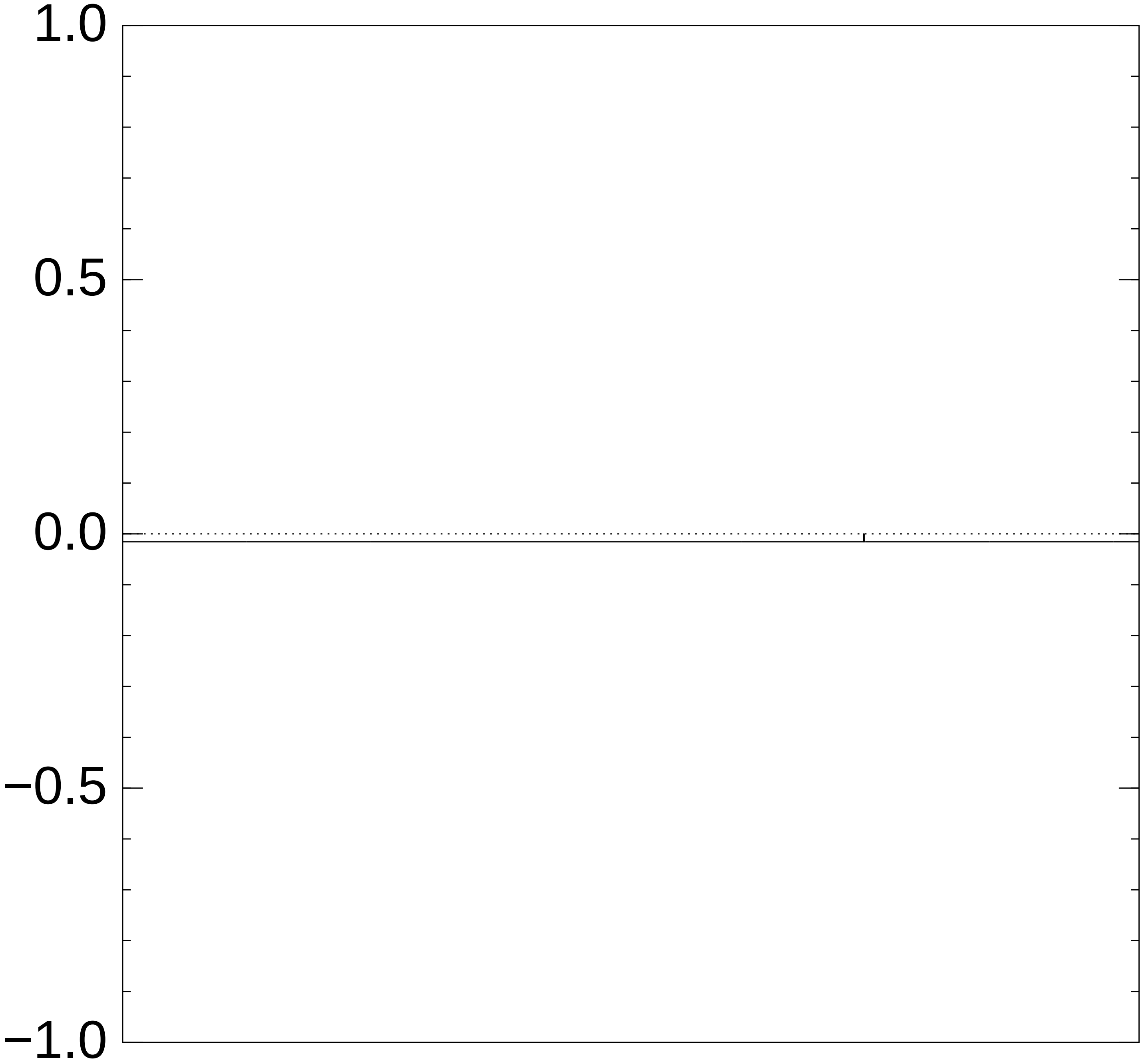
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $80 \times$ (Step $1 +$ Step 2):



Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $90 \times$ (Step $1 +$ Step 2):



Normalized graph of $u \mapsto a_u$ for an example with $n = 12$ after $100 \times (\text{Step } 1 + \text{Step } 2)$:



Very bad stopping point.

$u \mapsto a_u$ is completely described by a vector of two numbers (with fixed multiplicities):
(1) a_u for roots u;
(2) a_u for non-roots u.

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Easily compute eigenvalues
and powers of this linear map
to understand evolution
of state of Grover's algorithm.
$\Rightarrow$ Probability is $\approx$1
after $\approx(\pi/4)2^{0.5n}$ iterations.

Many more applications

Shor generalizations:
e.g., poly-time attack breaking "cyclotomic" case of Gentry STOC 2009 "Fully homomorphic encryption using ideal lattices".

Grover generalizations:
e.g., fastest subset-sum attacks use "quantum walks".

Not just Shor and Grover:
e.g., subexponential-time CRS/CSIDH isogeny attack uses "Kuperberg's algorithm".