Generating random primes faster

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cr.yp.to/papers.html#pqrsa

The standard algorithm
to generate random primes:

```
proof.arithmetic(False)
while True:
  p = randrange(2^(n-1),2^n)
  p = ZZ(p)
  if p.is_prime(): print p
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Standard speedup using wheels:
e.g., force $p \bmod 6 \in \{1,5\}$.
Wheel using all primes $q \le n^{O(1)}$:
$n^{1+o(1)}$ iterations per prime.
Recall $\prod_{q\le y}\left(1-\frac{1}{q}\right) \in \Theta\left(\frac{1}{\log y}\right)$.

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Most iterations are much simpler: Fermat test rejects p. Fast reject by trial division/ECM? Still $n^{3+o(1)}$ bit ops per prime.

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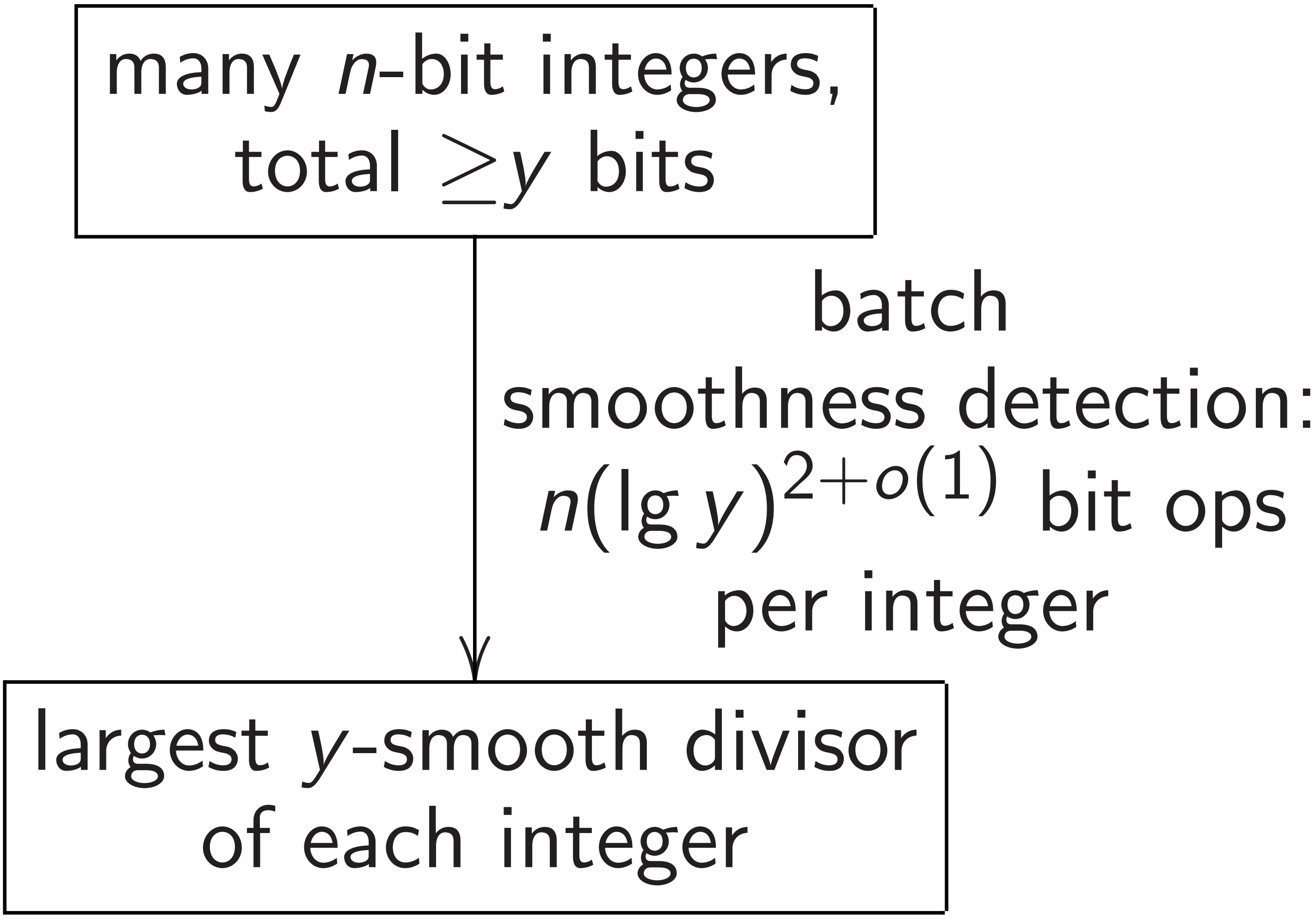
Recall:



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Recall:

many n-bit integers, total $\geq y$ bits

↓ batch smoothness detection: $n(\lg y)^{2+o(1)}$ bit ops per integer

largest y-smooth divisor of each integer

Apply batch smoothness detection for $y = 2^{2^0}$, then $y = 2^{2^1}$, then $y = 2^{2^2}$, ..., then $y \approx 2^{n^{0.5+o(1)}}$.