# Generating random primes faster

D. J. Bernstein

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pqRSA project team:

Daniel J. Bernstein
Josh Fried
Nadia Heninger
Paul Lou
Luke Valenta

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The standard algorithm
to generate random primes:

```
proof.arithmetic(False)
while True:
  p = randrange(2^(n-1),2^n)
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$n^{1+o(1)}$ iterations per prime.

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Standard speedup using wheels:
e.g., force $p \bmod 6 \in \{1, 5\}$.
Wheel using all primes $q \le n^{O(1)}$:
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Recall $\prod_{q \le y} \left(1 - \frac{1}{q}\right) \in \Theta\left(\frac{1}{\log y}\right)$.

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2010 Bernstein conjecture:
correctly recognize primality using
$n^{o(1)}$ tests, total $n^{2+o(1)}$ bit ops.
Fermat test, then Lucas test
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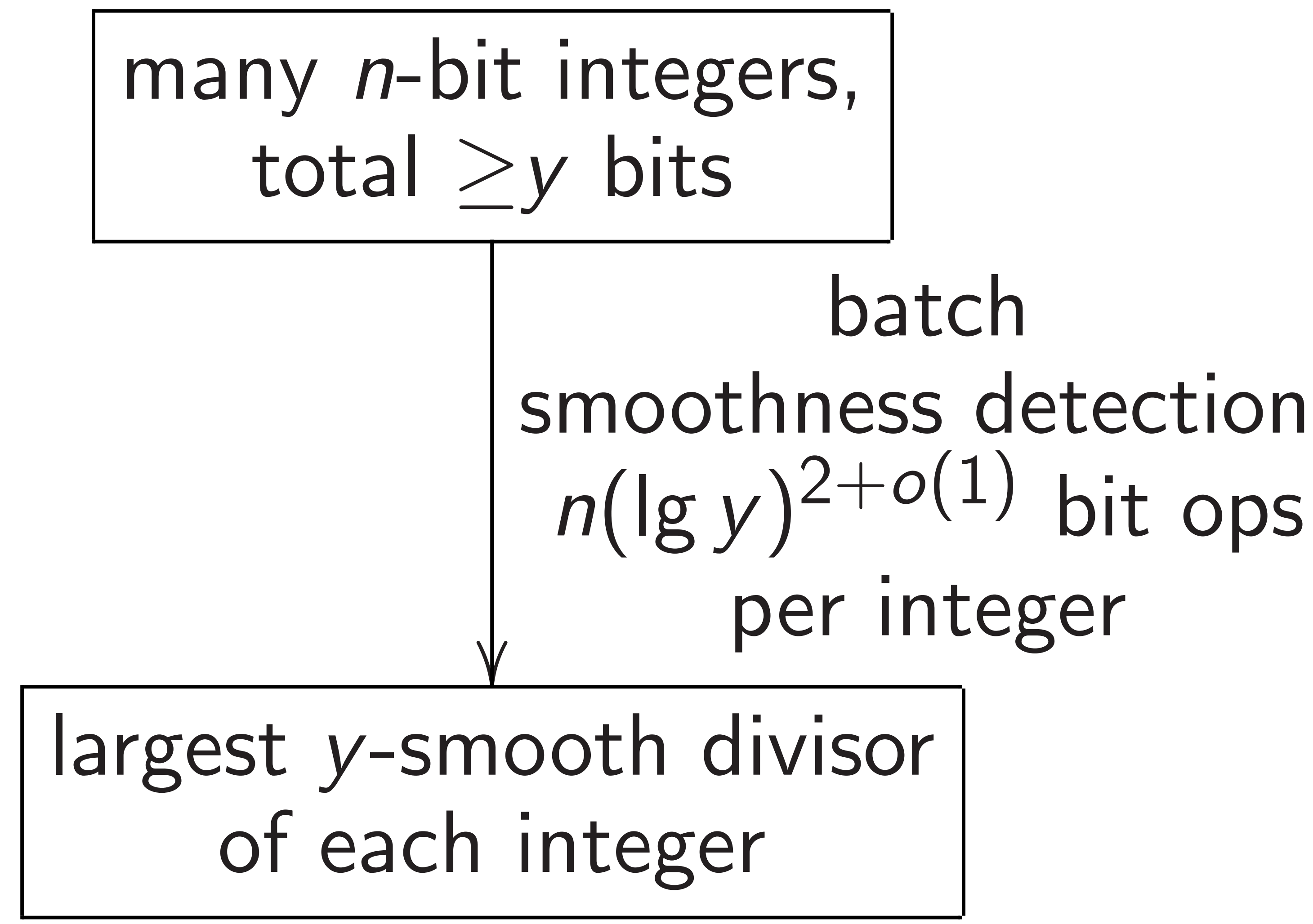

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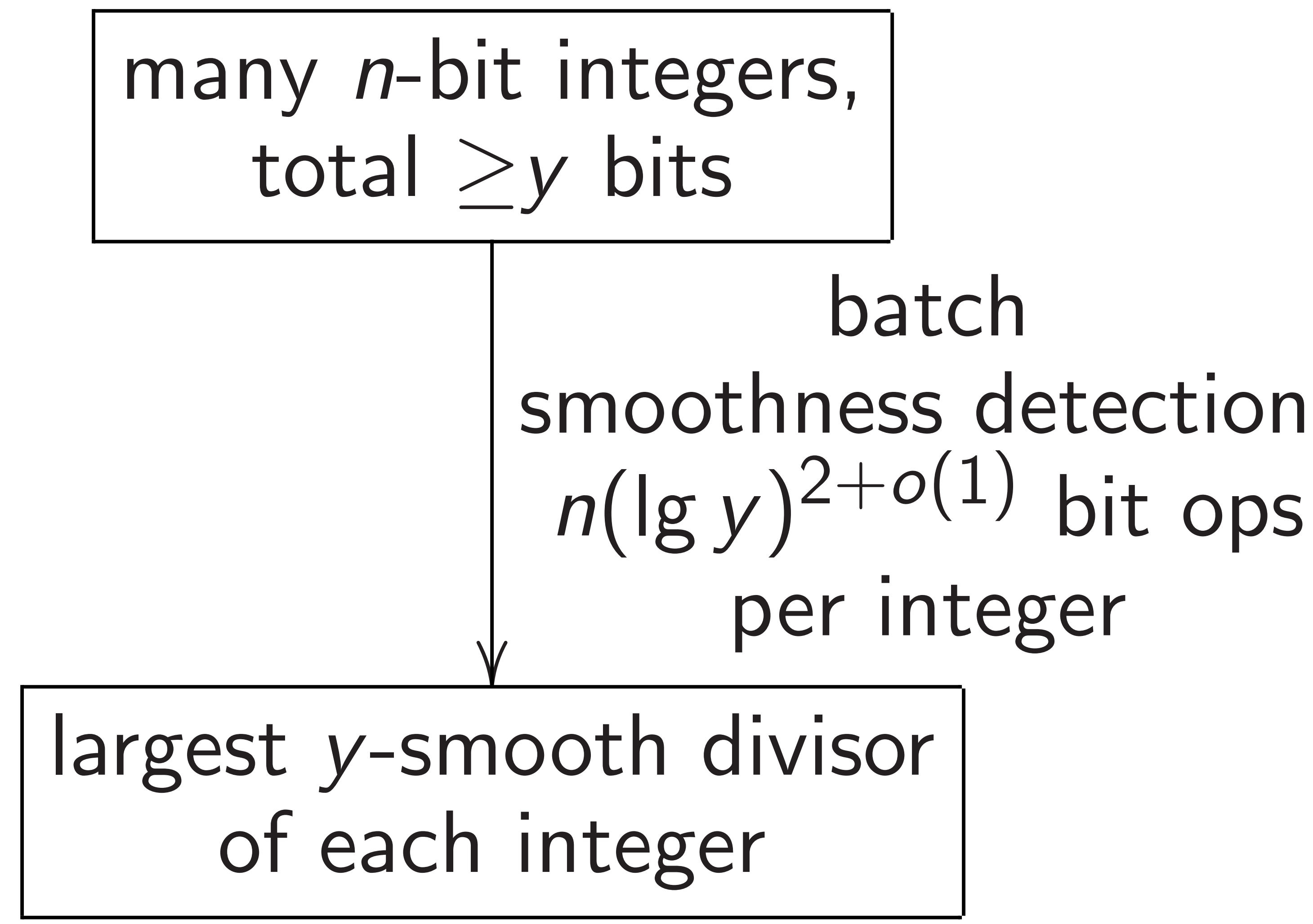


Apply batch smoothness detection for $y = 2^{2^0}$, then $y = 2^{2^1}$, then $y = 2^{2^2}$, . . . , then $y \approx 2^{n^{0.5+o(1)}}$.