How to multiply big integers

Standard idea: Use polynomial with coefficients in $\{0, 1, \ldots, 9\}$ to represent integer in radix 10.

Example of representation:
$839 = 8 \cdot 10^2 + 3 \cdot 10^1 + 9 \cdot 10^0 =$ value (at $t = 10$) of polynomial $8t^2 + 3t^1 + 9t^0$.

Convenient to express polynomial inside computer as array $9, 3, 8$ (or $9, 3, 8, 0$ or $9, 3, 8, 0, 0$ or $\ldots$): "`p[0] = 9; p[1] = 3; p[2] = 8`"

Multiply two integers by multiplying polynomials that represent the integers.

Polynomial multiplication involves *small* integer coefficients. Have split one big multiplication into many small operations.

Example, squaring 839:

$(8t^2 + 3t^1 + 9t^0)^2 =$
$8t^2(8t^2 + 3t^1 + 9t^0) +$
$3t^1(8t^2 + 3t^1 + 9t^0) +$
$9t^0(8t^2 + 3t^1 + 9t^0) =$
$64t^4 + 48t^3 + 153t^2 + 54t^1 + 81t^0$.

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$70t^4 + 3$

$7t^5 + 0t$

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$64t^4 + 48t^3 + 159$

$64t^4 + 63t^3 + 9t^2$

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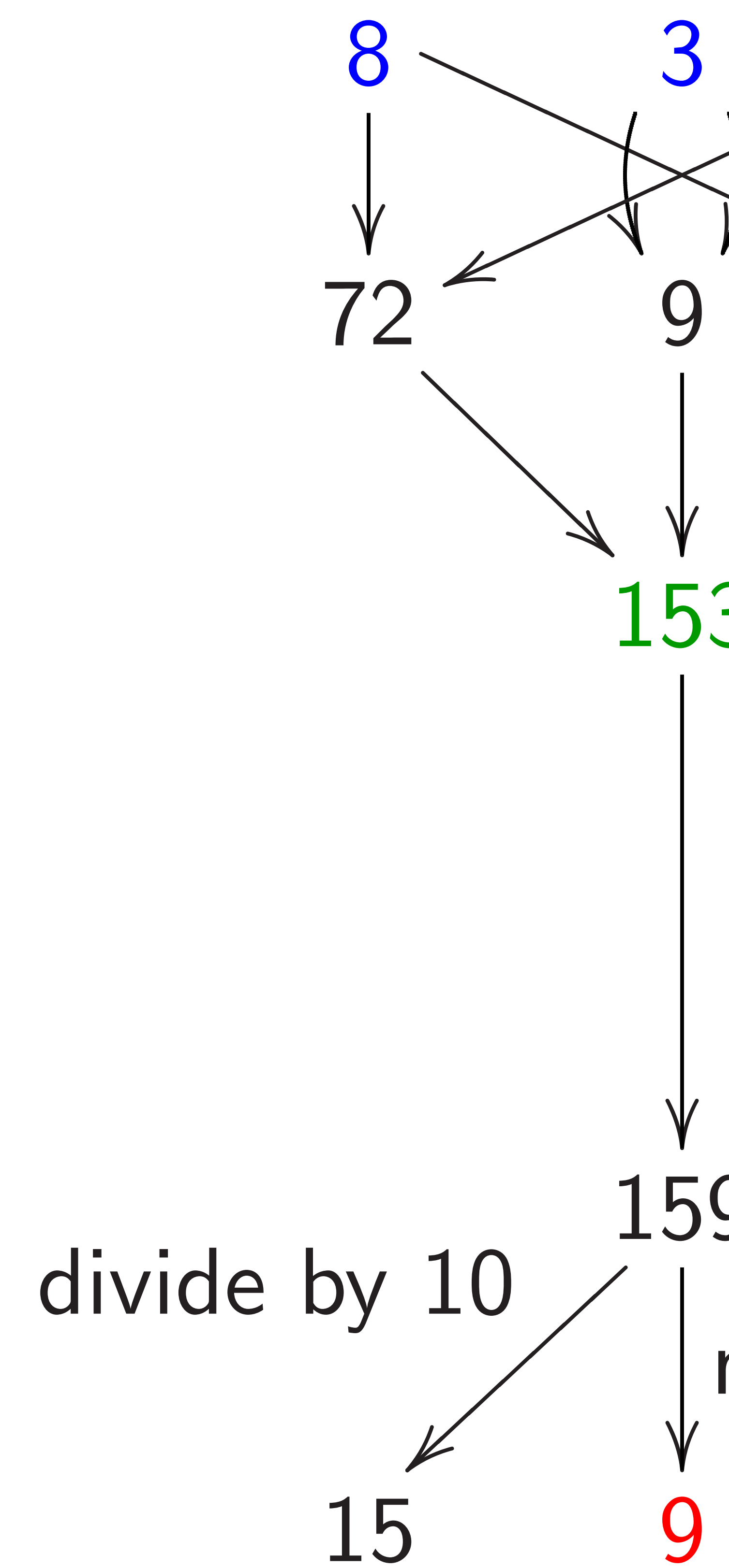

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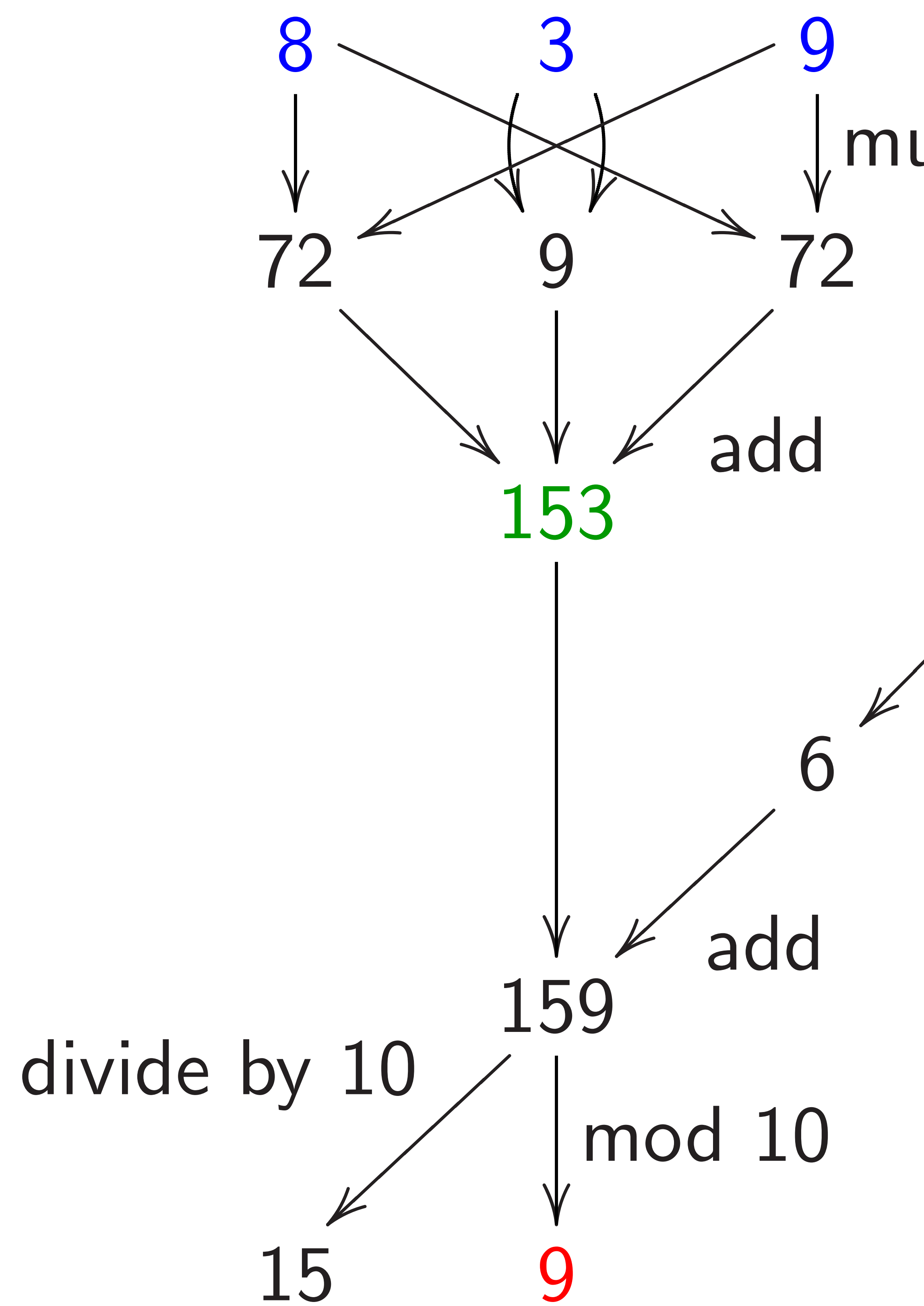

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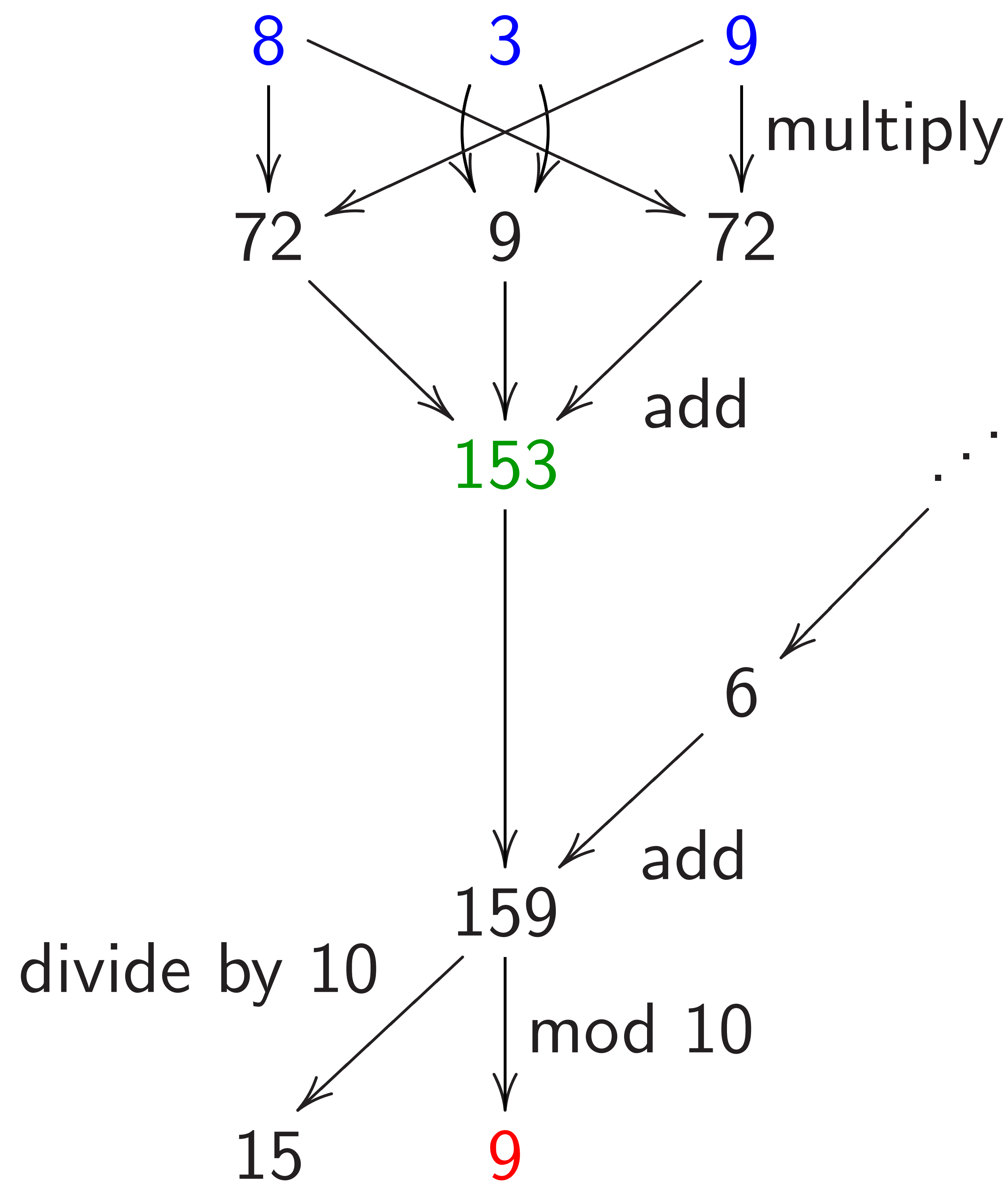

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$63t^3 + 9t^2 + 2t^1 + 1t^0$;

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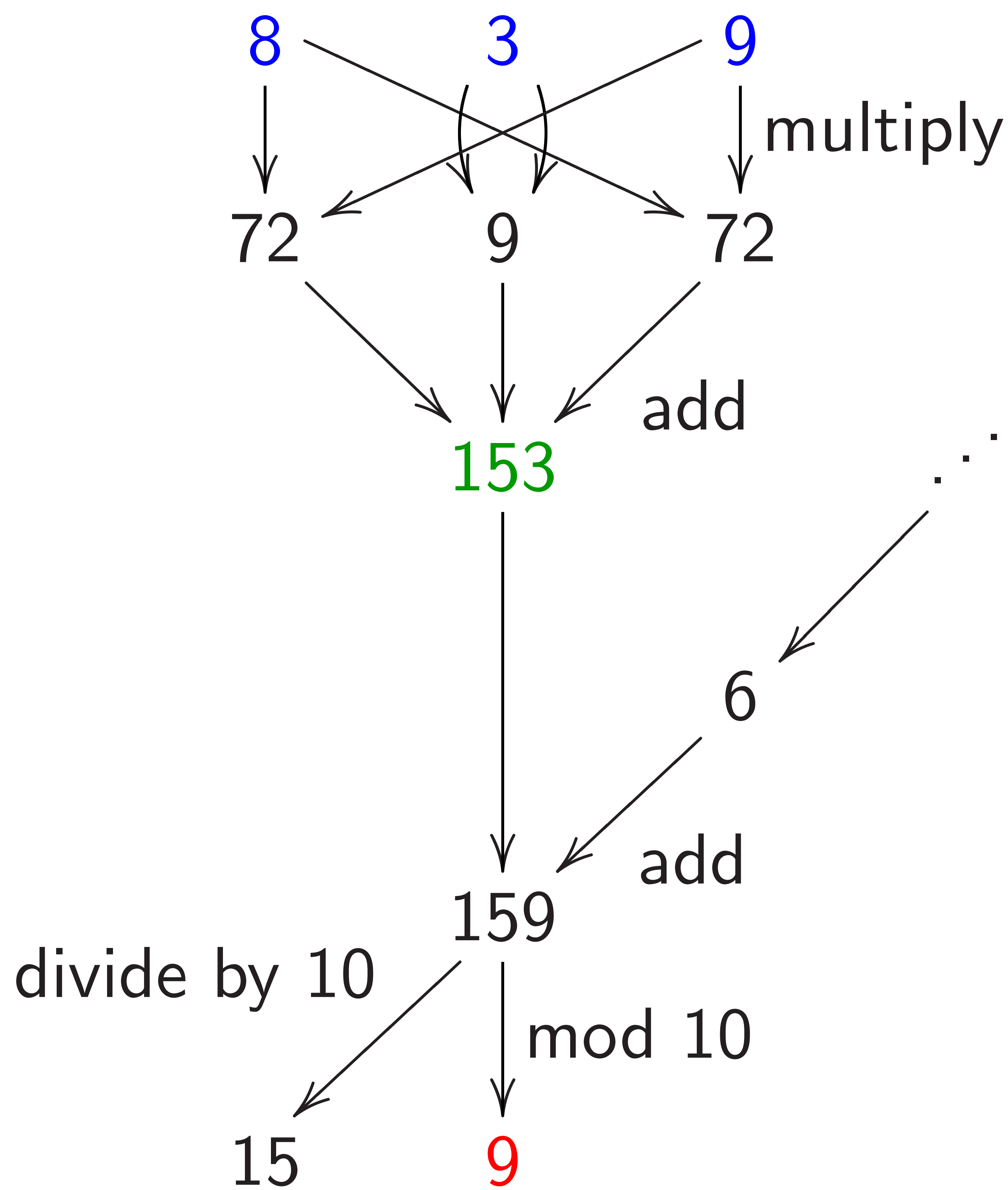

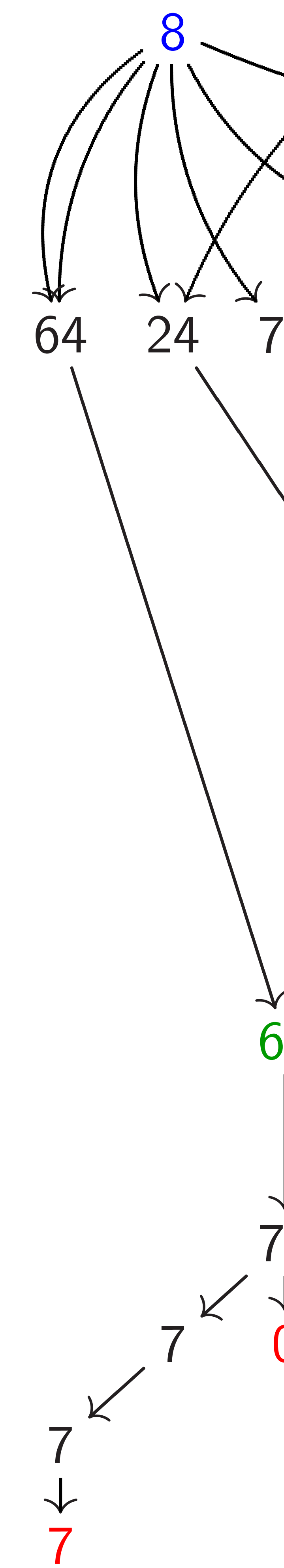

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839:

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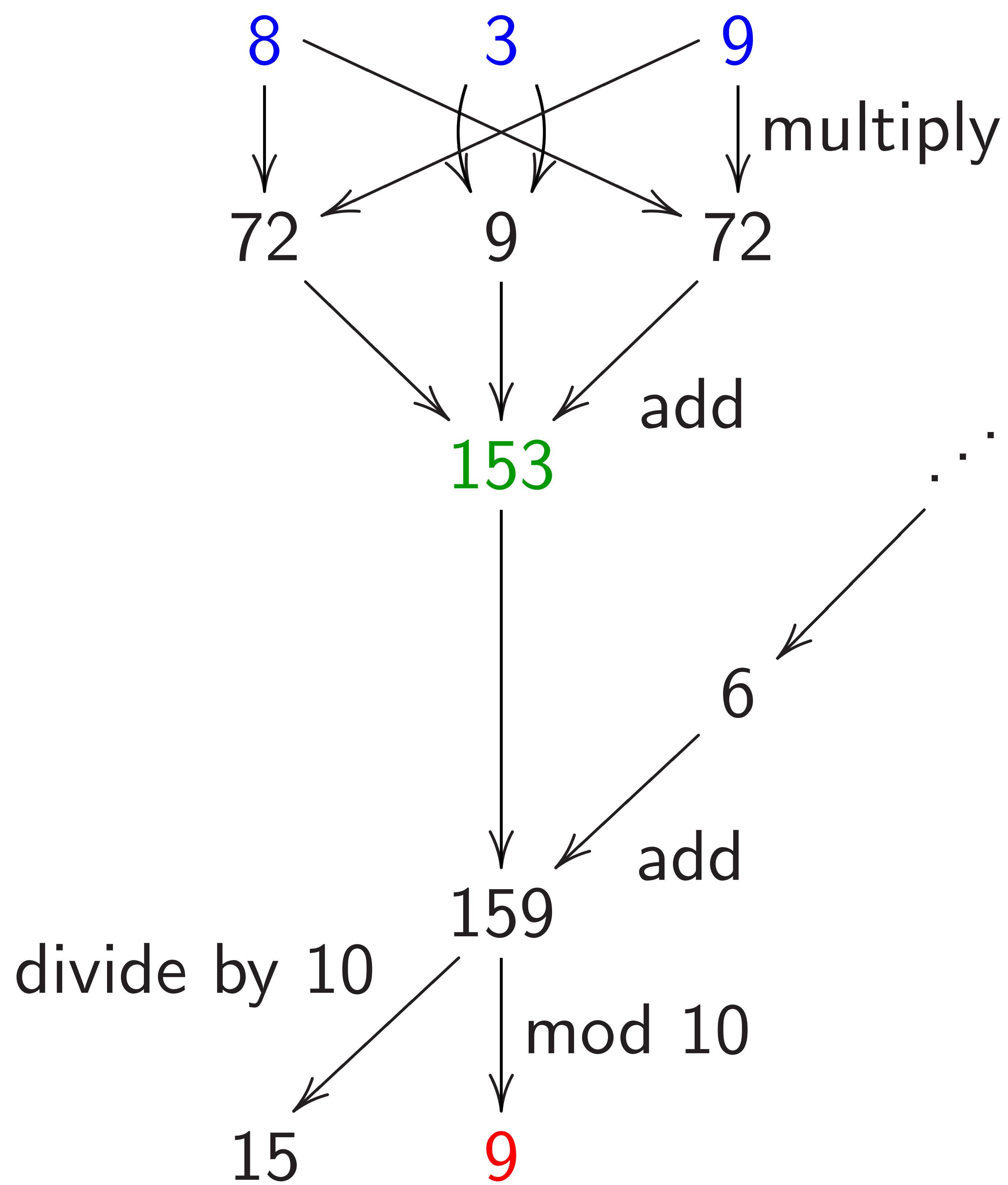

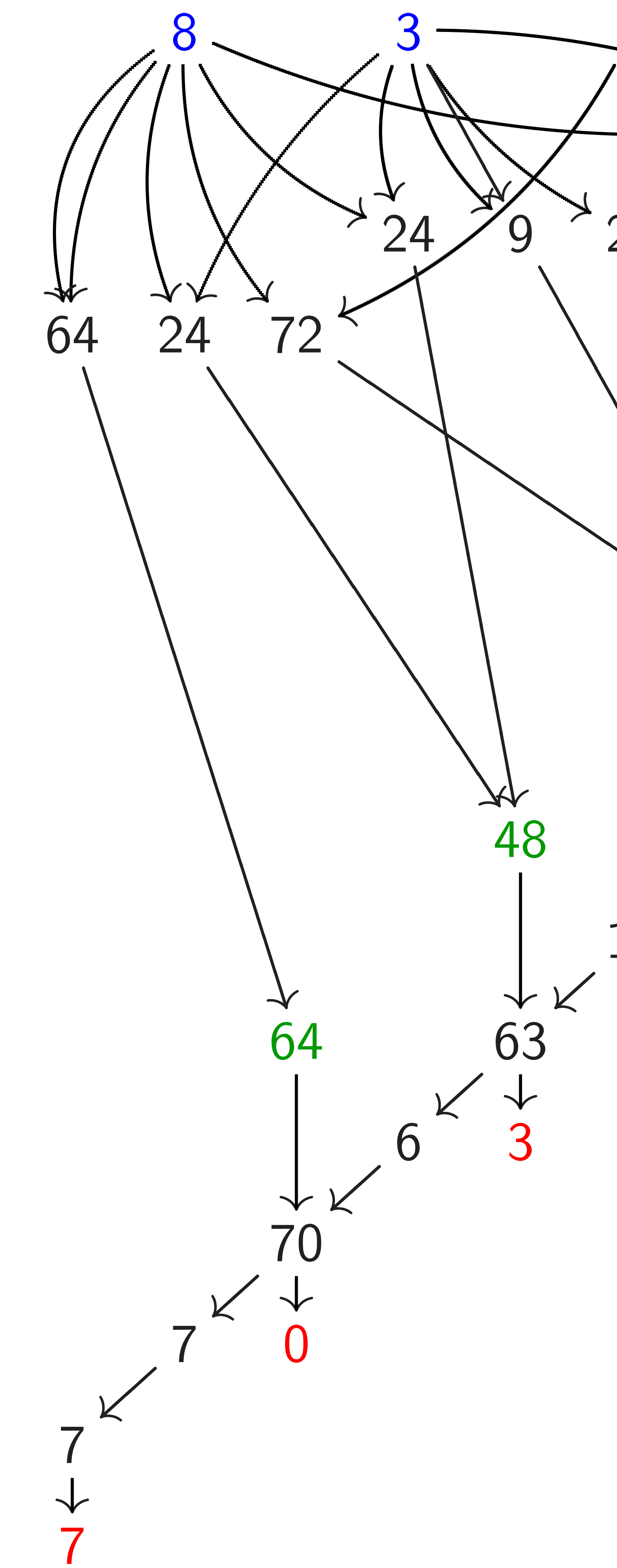

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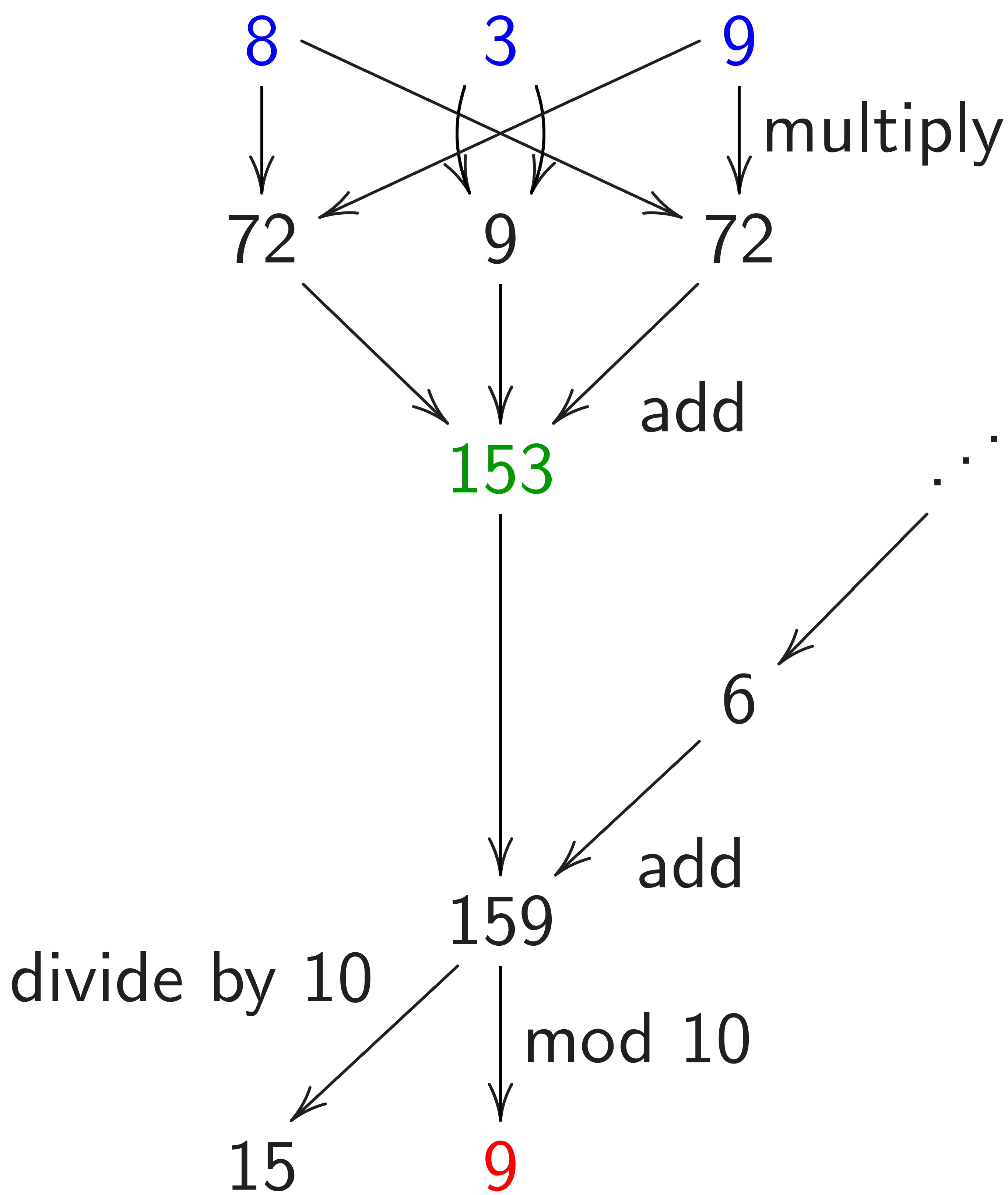

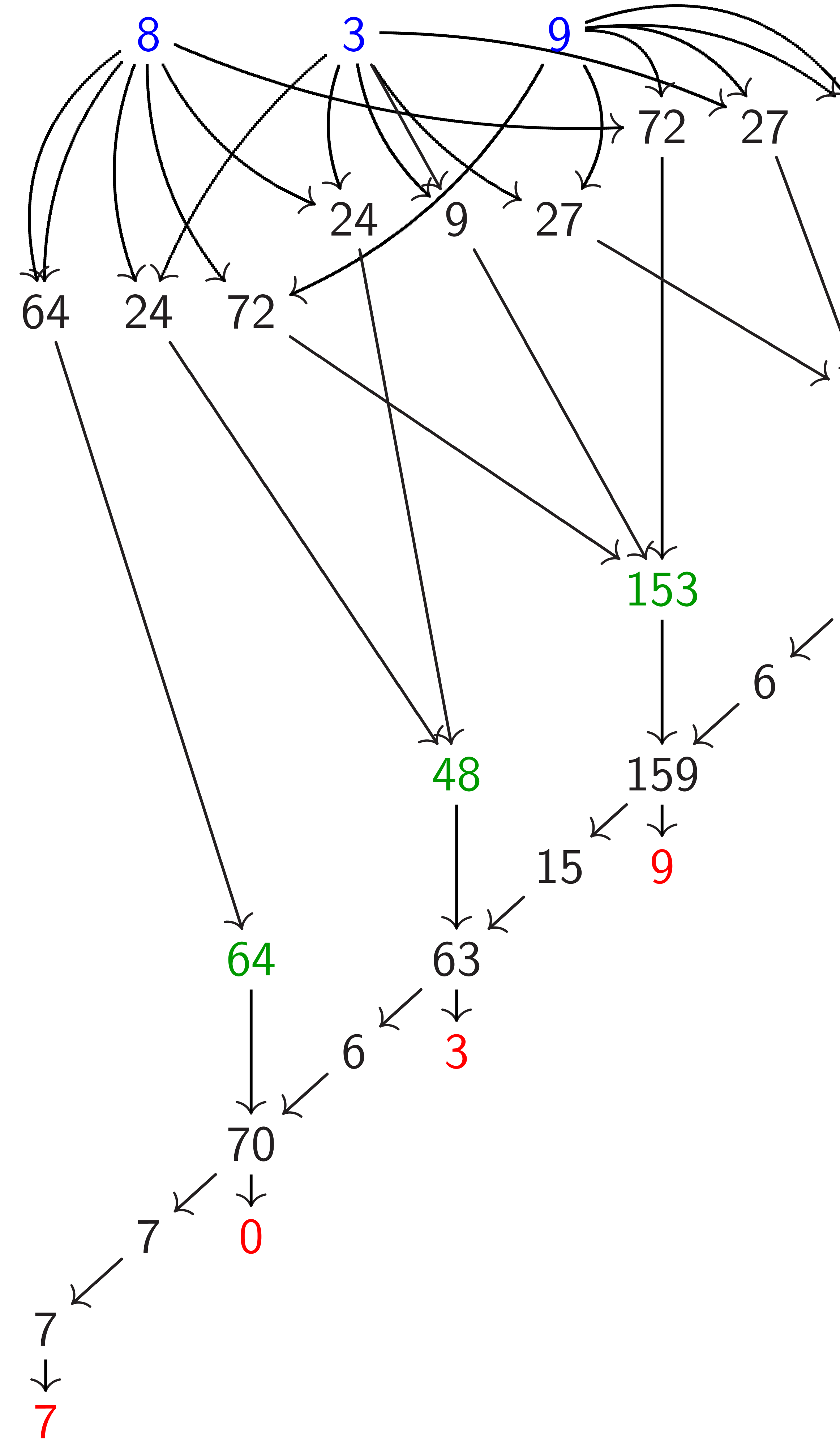

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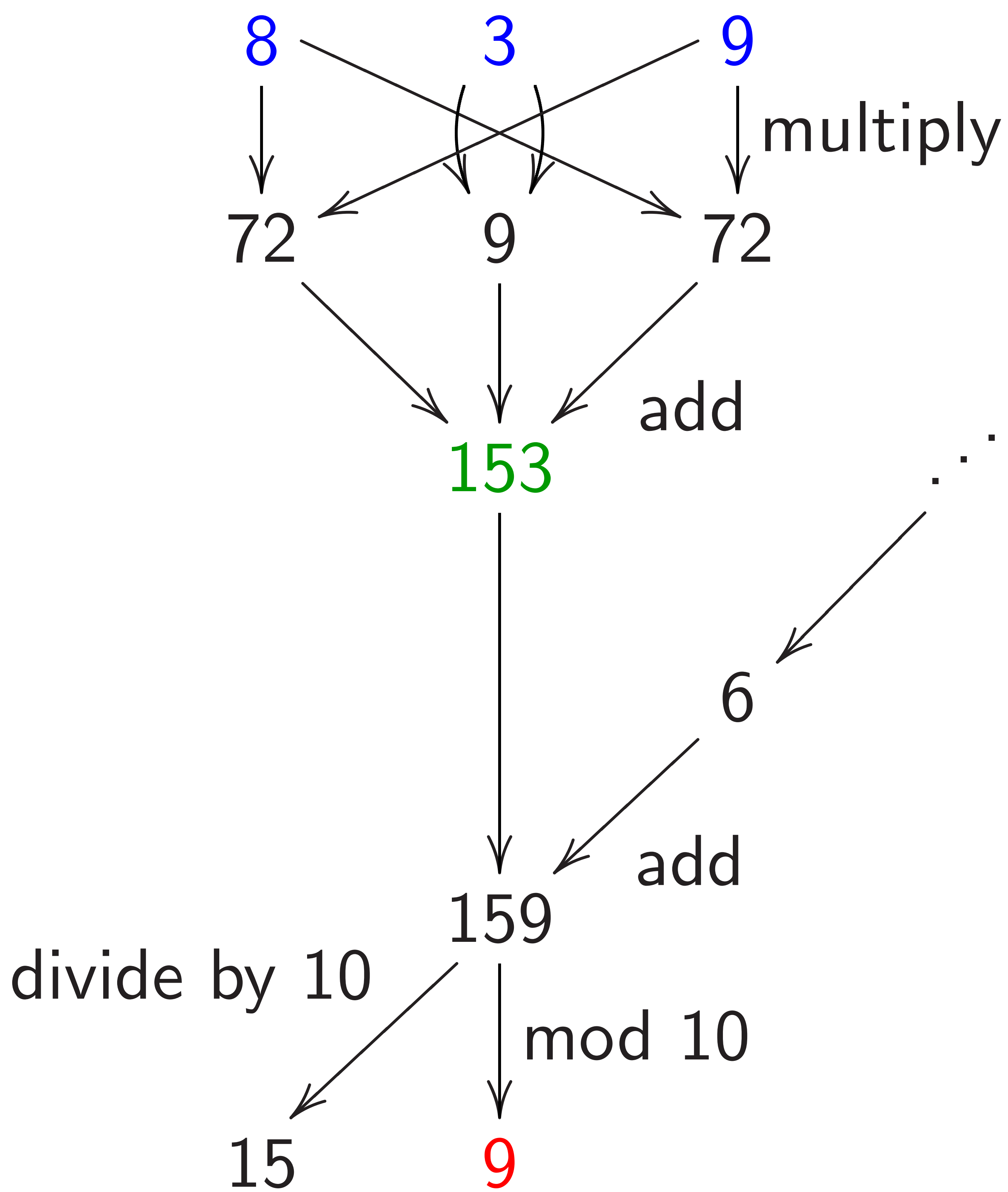

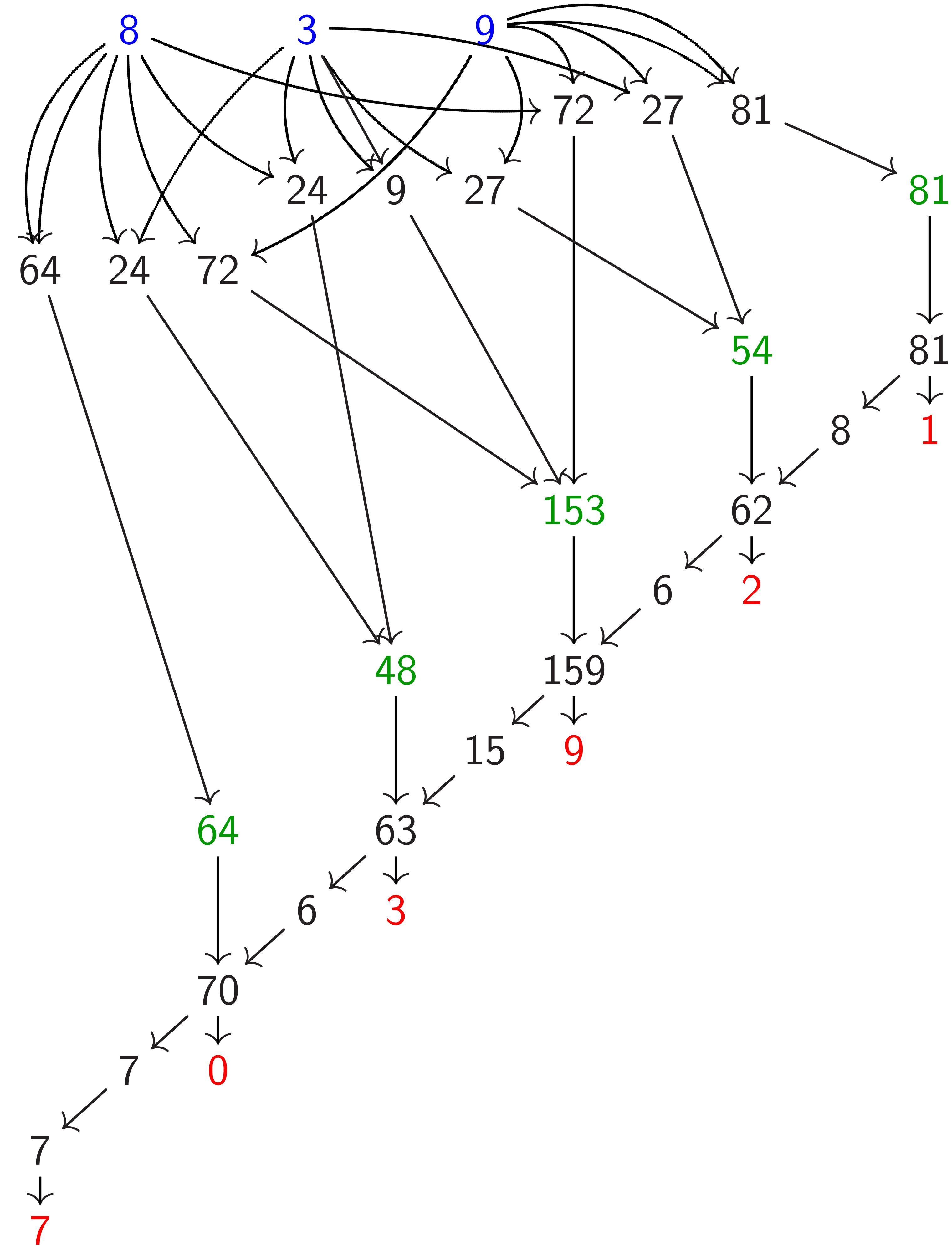

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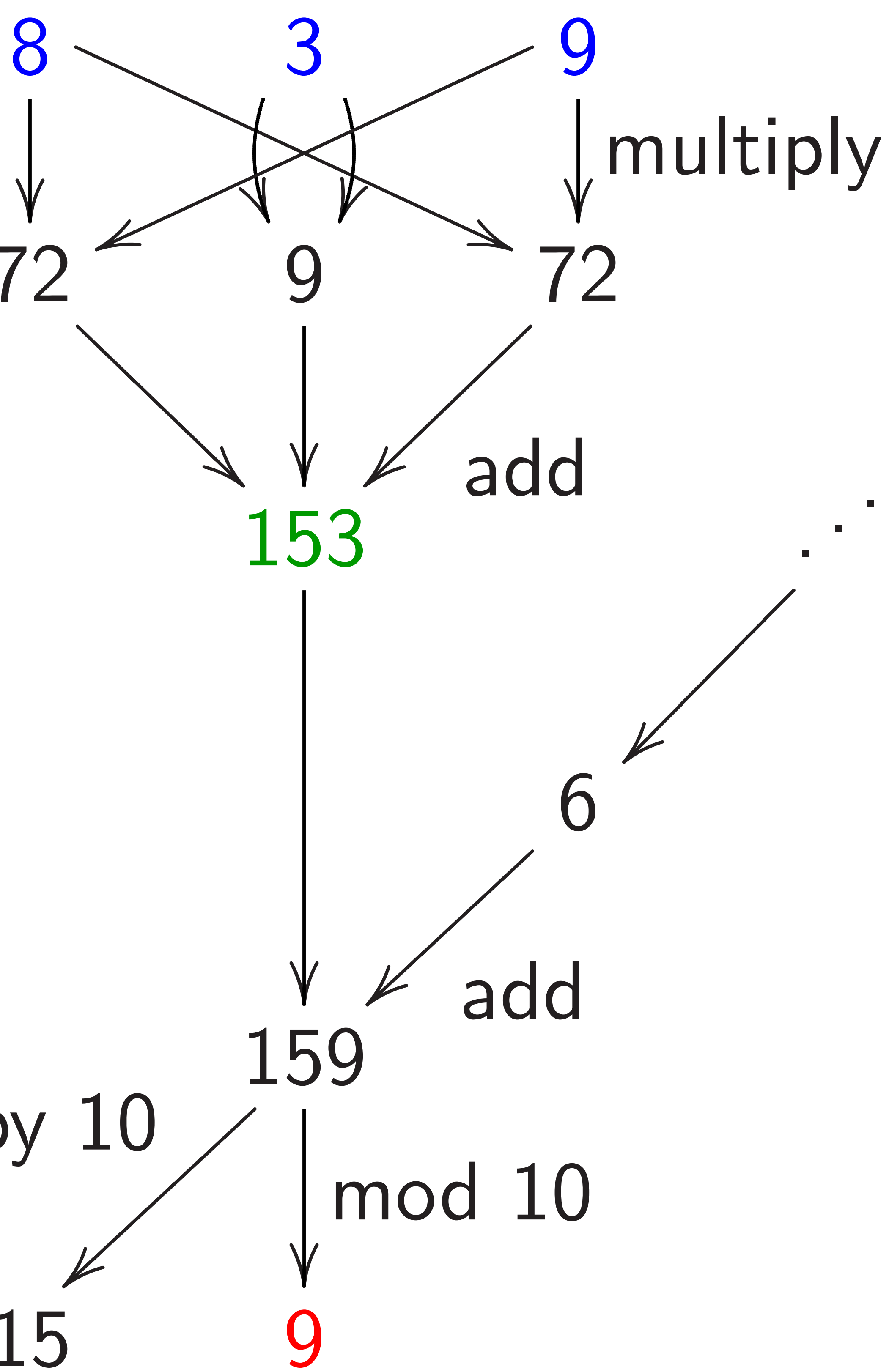
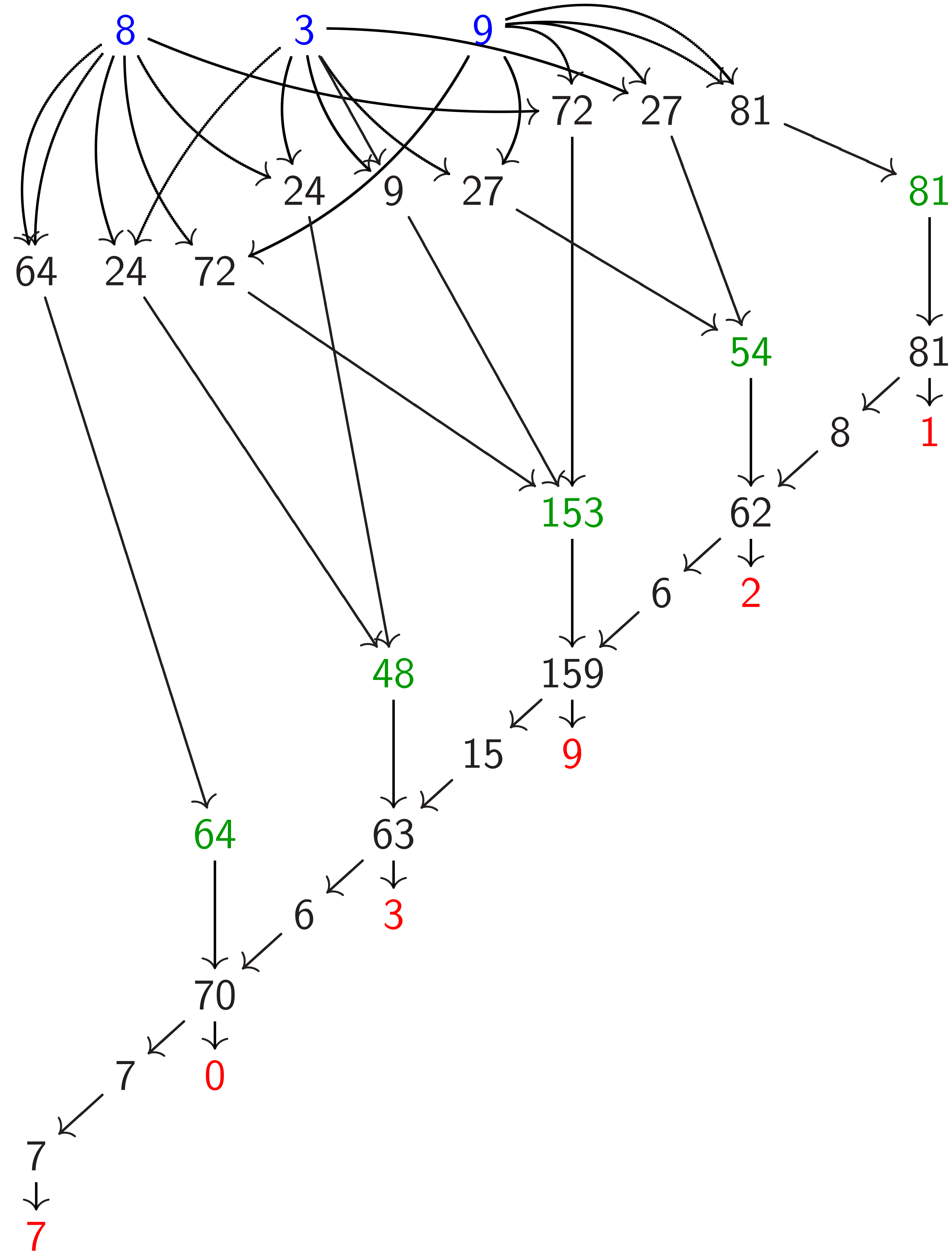




The scal

$839 = 8$

value (a

$800t^2 +$

Squaring

$640000t$

$540t^1 +$

Carrying

$640000t$

$540t^1 +$

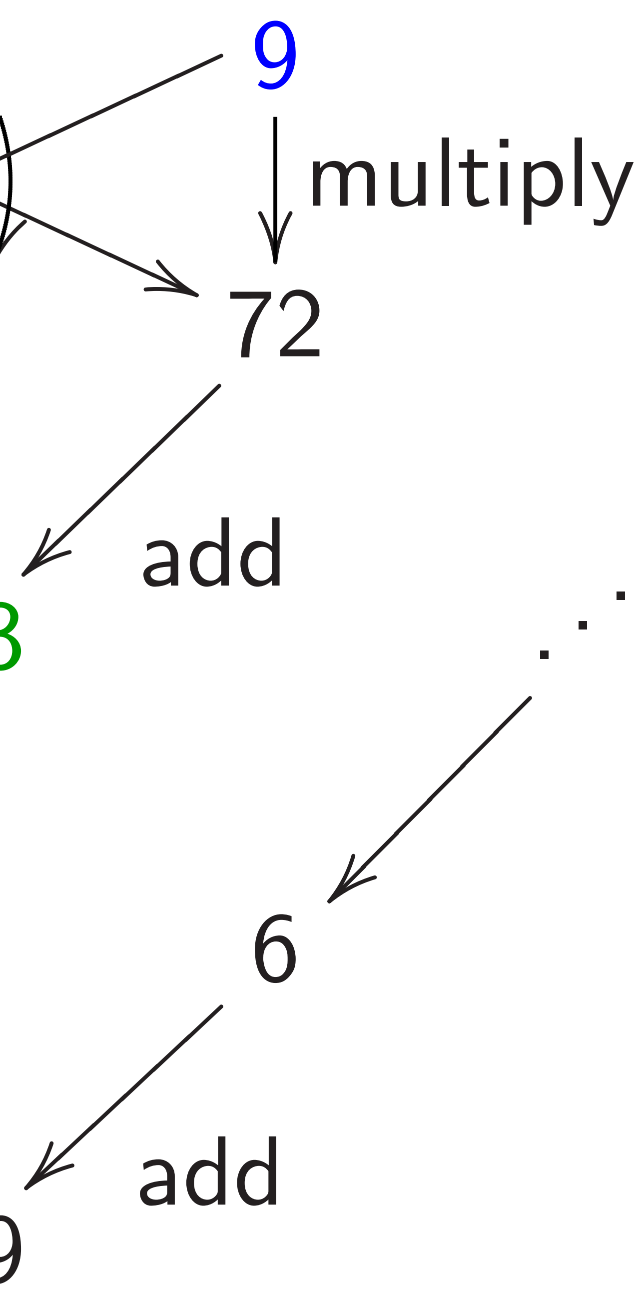
$640000t$

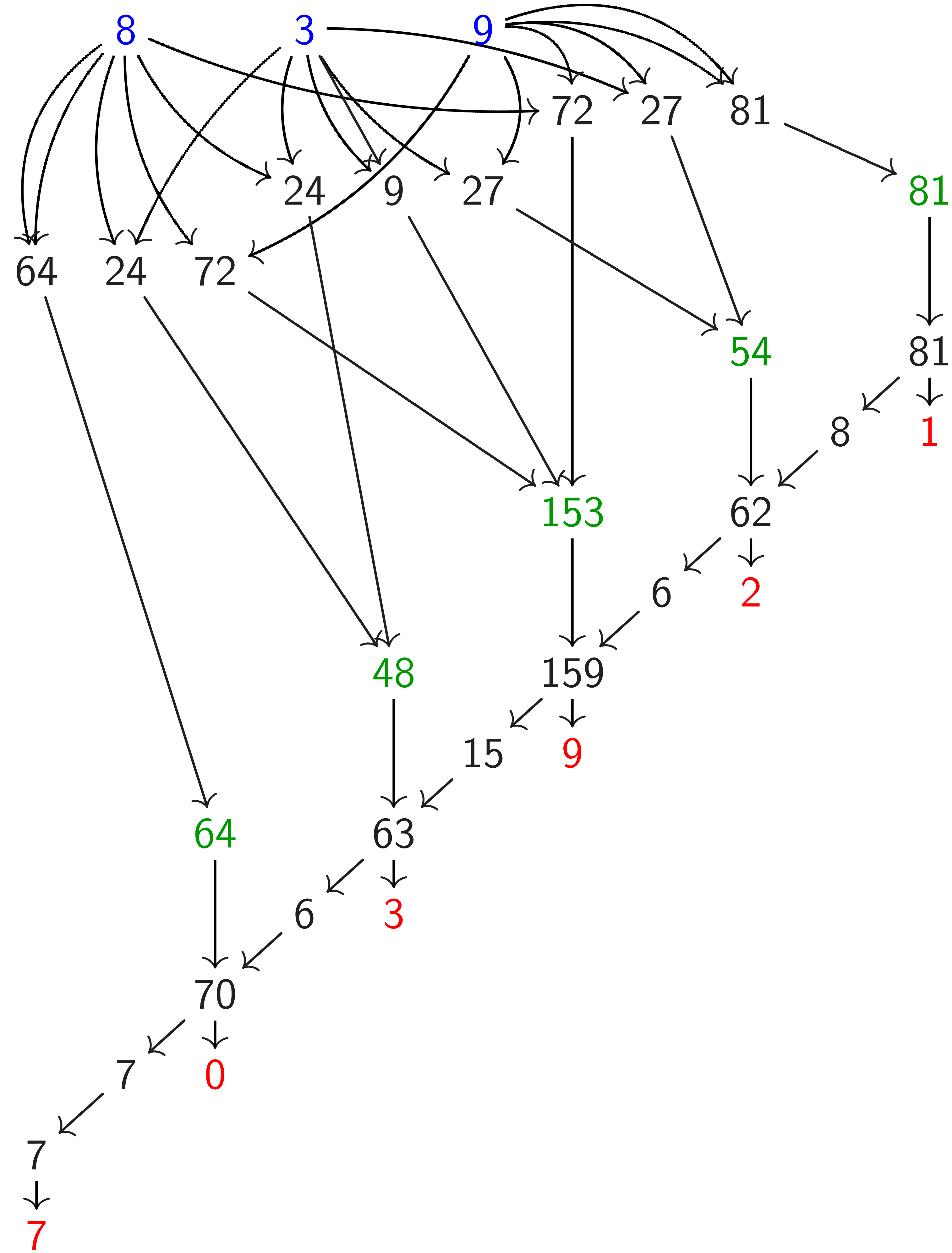
$620t^1 +$

$700000t$

$20t^1 + 1$

were used here?





The scaled variatio

$839 = 800 + 30 +$

value (at $t = 1$) o

$800t^2 + 30t^1 + 9t$

Squaring: $(800t^2$

$640000t^4 + 48000$

$540t^1 + 81t^0$.

Carrying:

$640000t^4 + 48000$

$540t^1 + 81t^0$;

$640000t^4 + 48000$

$620t^1 + 1t^0$; .

$700000t^5 + 0t^4 + 3$

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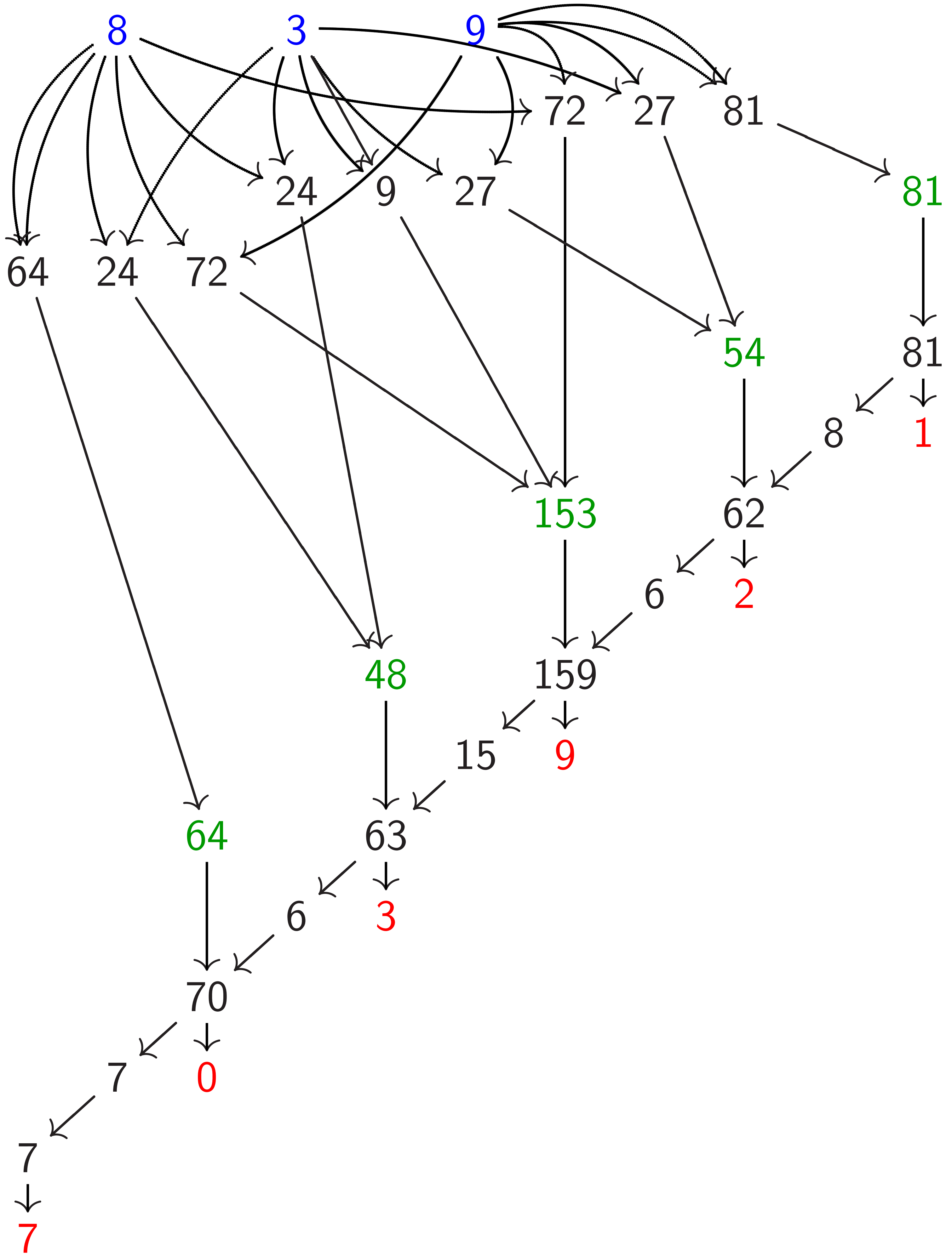

The scaled variation

$839 = 800 + 30 + 9 =$

value (at $t = 1$) of polynom

$800t^2 + 30t^1 + 9t^0$.

Squaring: $(800t^2 + 30t^1 + 9t$

$640000t^4 + 48000t^3 + 1530$

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Carrying:

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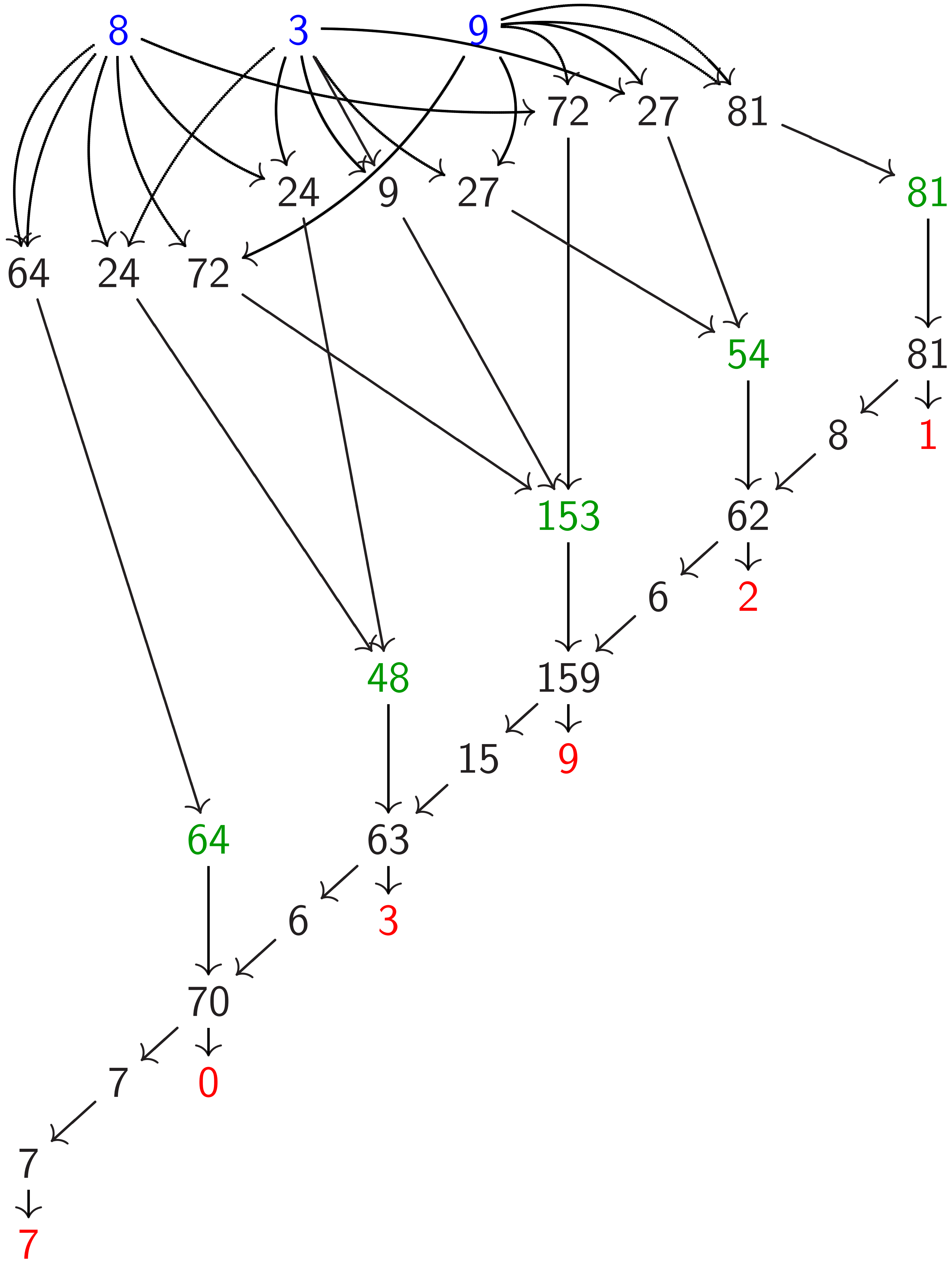

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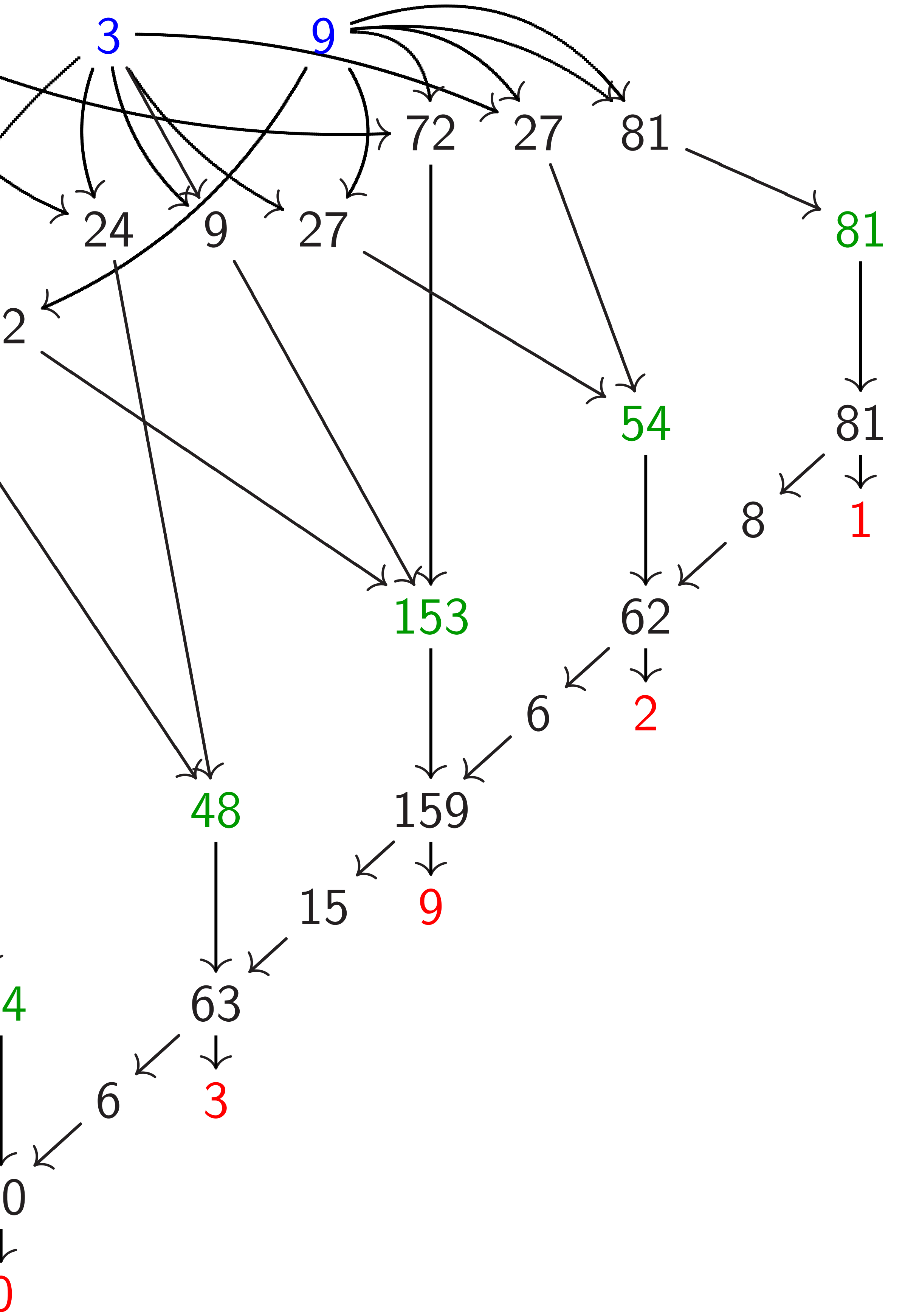

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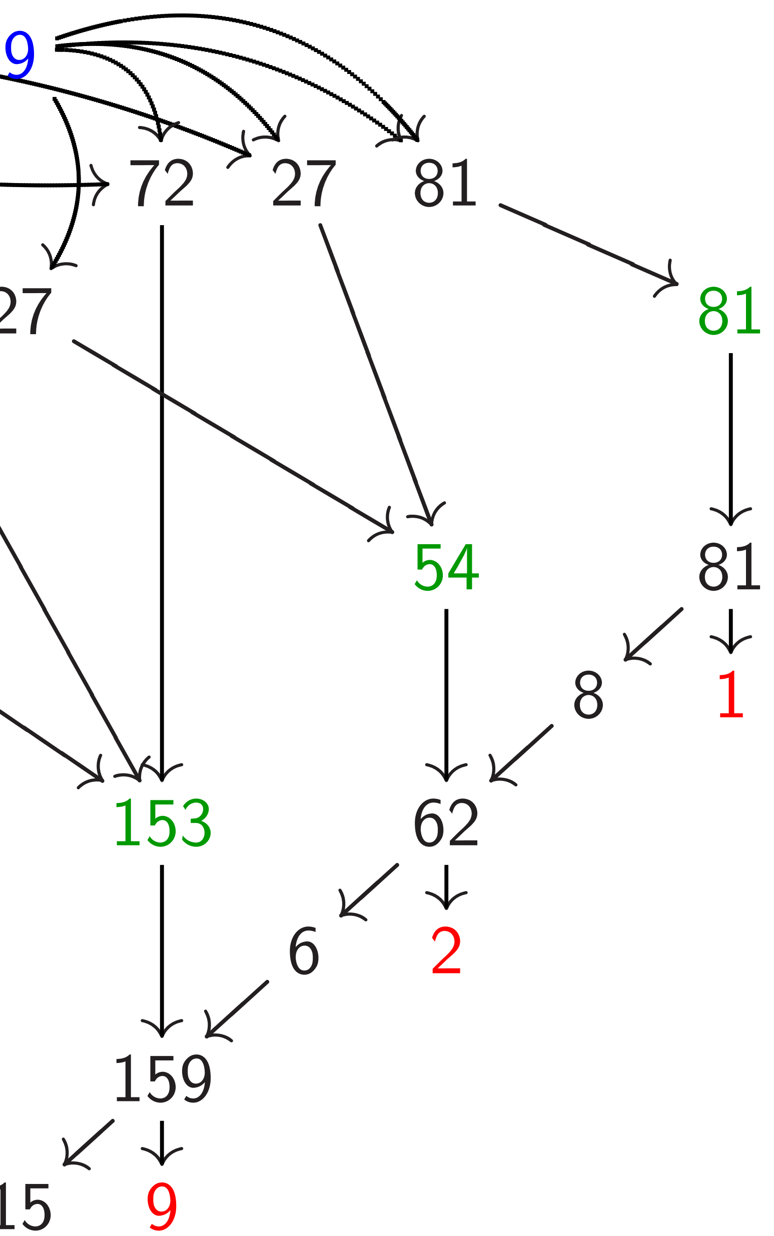

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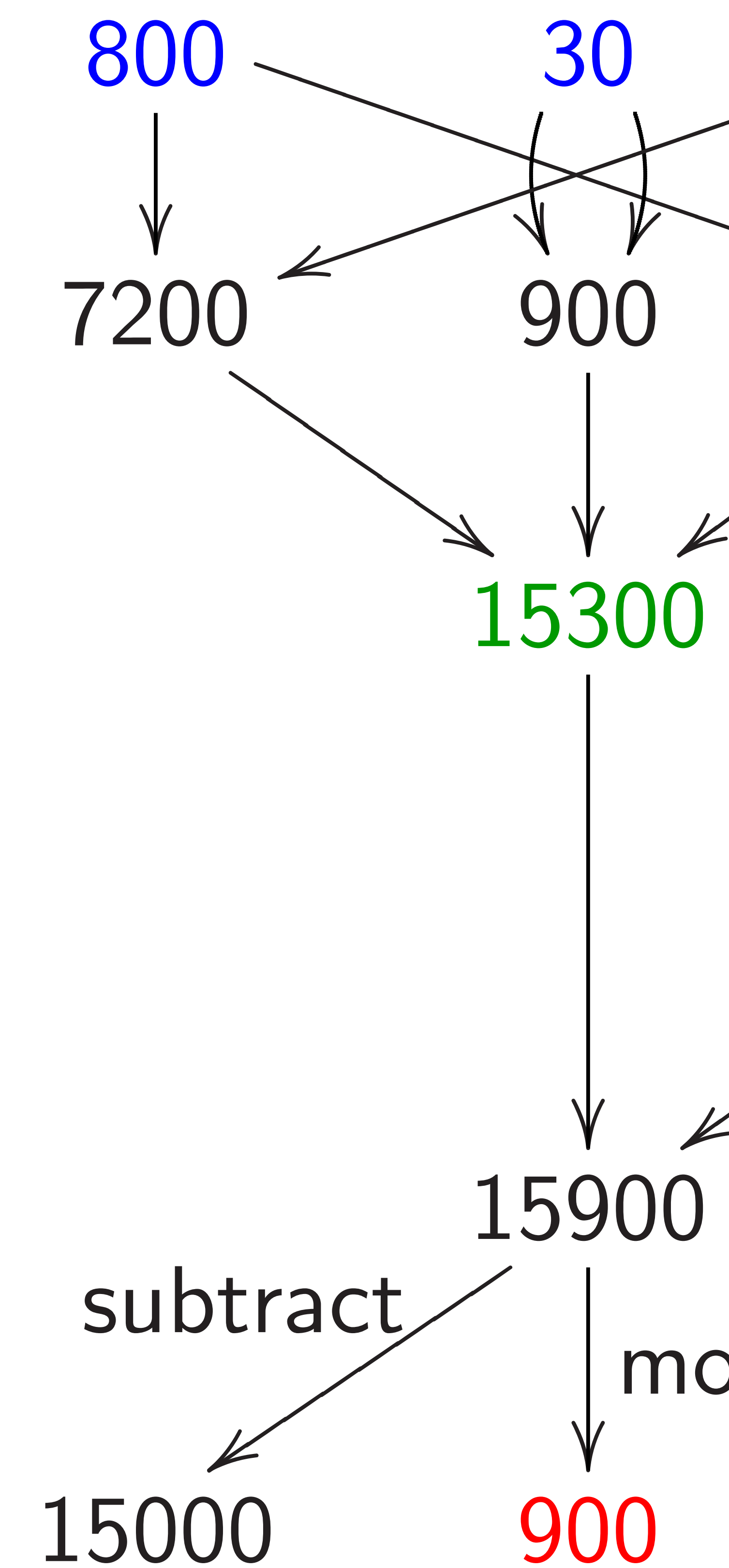

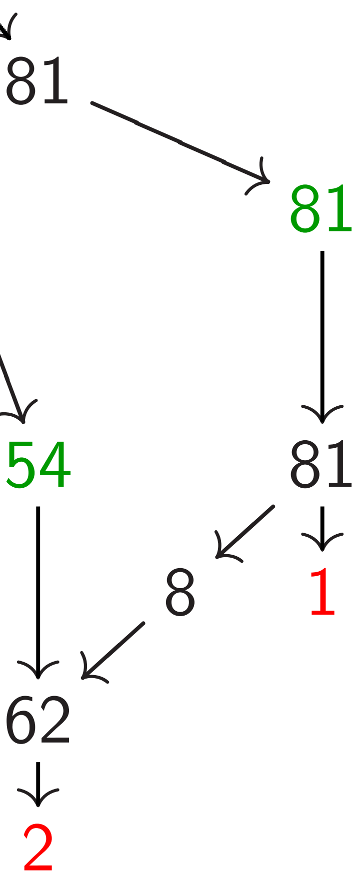

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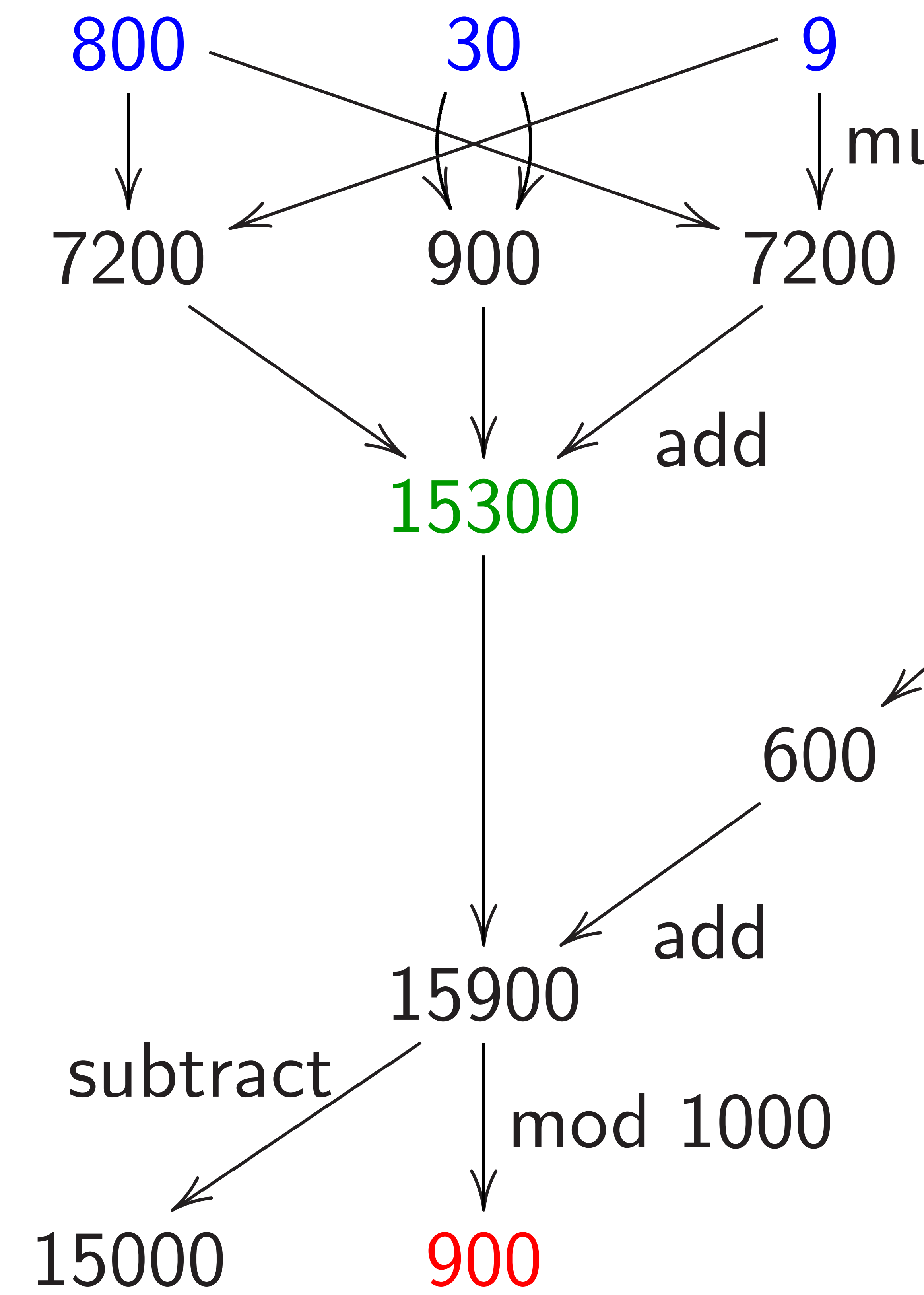

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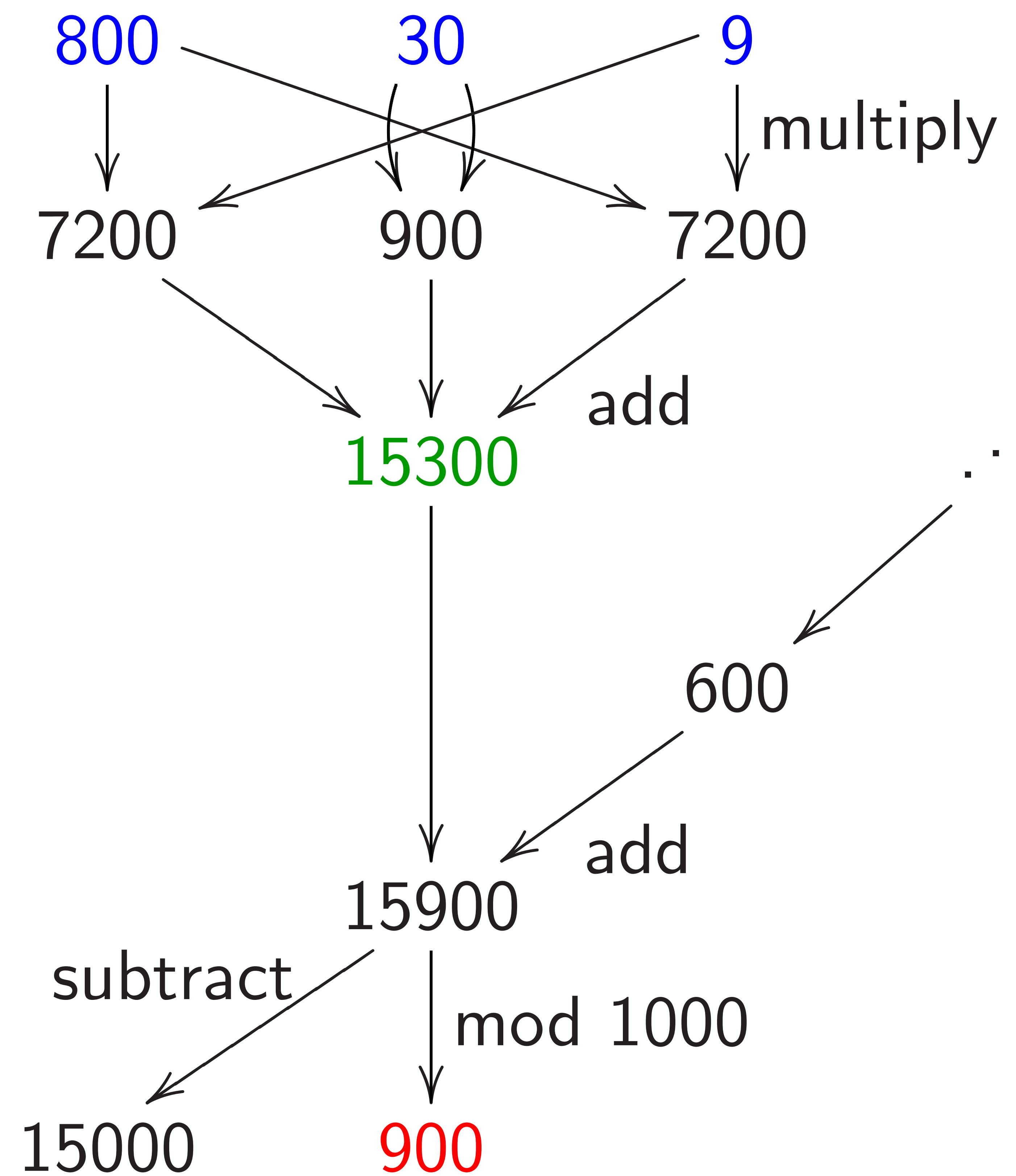

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$00 + 30 + 9 =$

t $t = 1$) of polynomial

$30t^1 + 9t^0$.

g: $(800t^2+30t^1+9t^0)^2 =$

$^4 + 48000t^3 + 15300t^2 +$

$81t^0$.

:

$^4 + 48000t^3 + 15300t^2 +$

$81t^0$;

$^4 + 48000t^3 + 15300t^2 +$

$1t^0$; . . .

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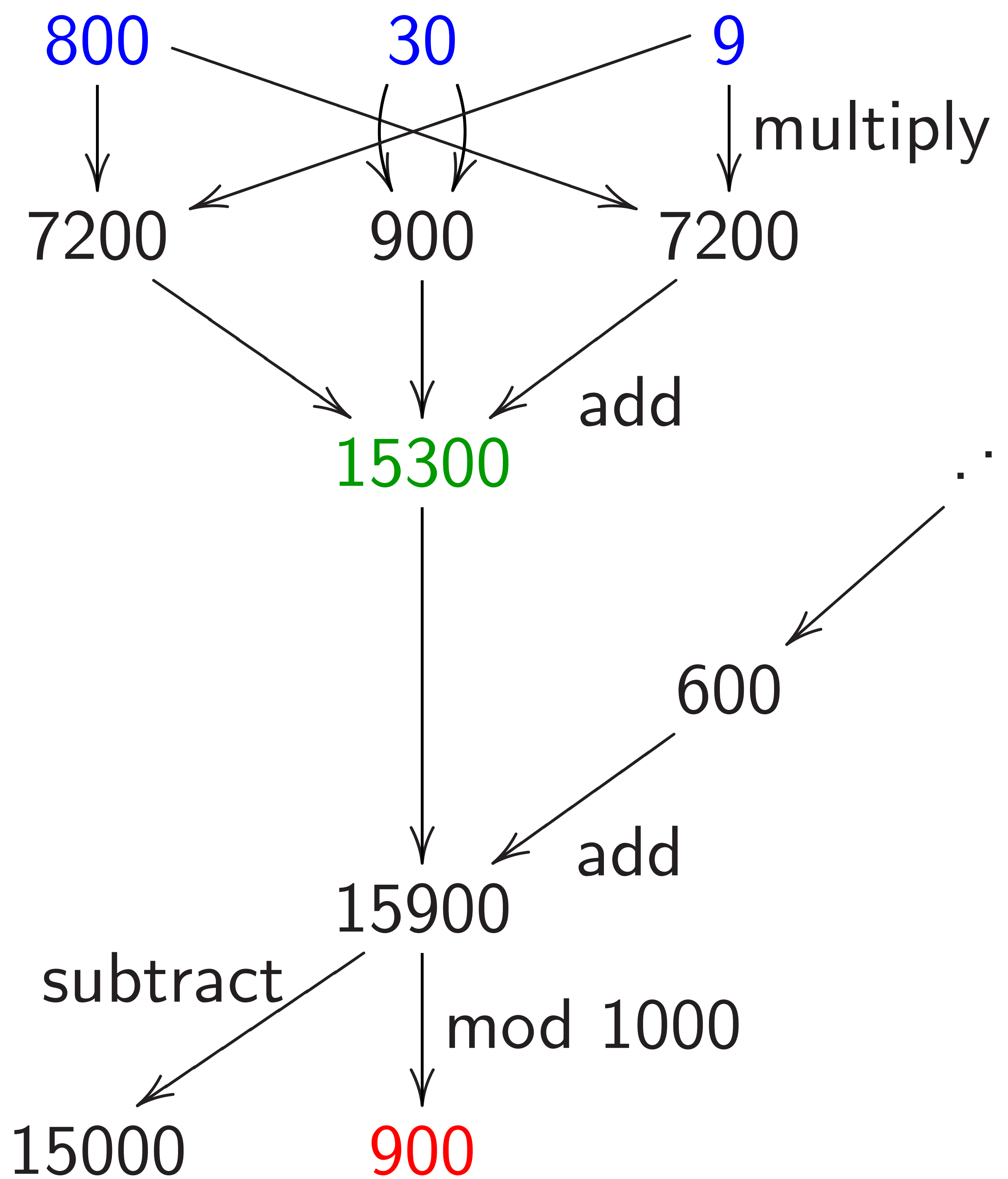


Speedup

$(\cdots + f_2$

has coef

$f_4 f_0 + f_3$

5 mults,

on

9 =

f polynomial

0.

$+30t^1+9t^0)^2 =$

$t^3 + 15300t^2 +$

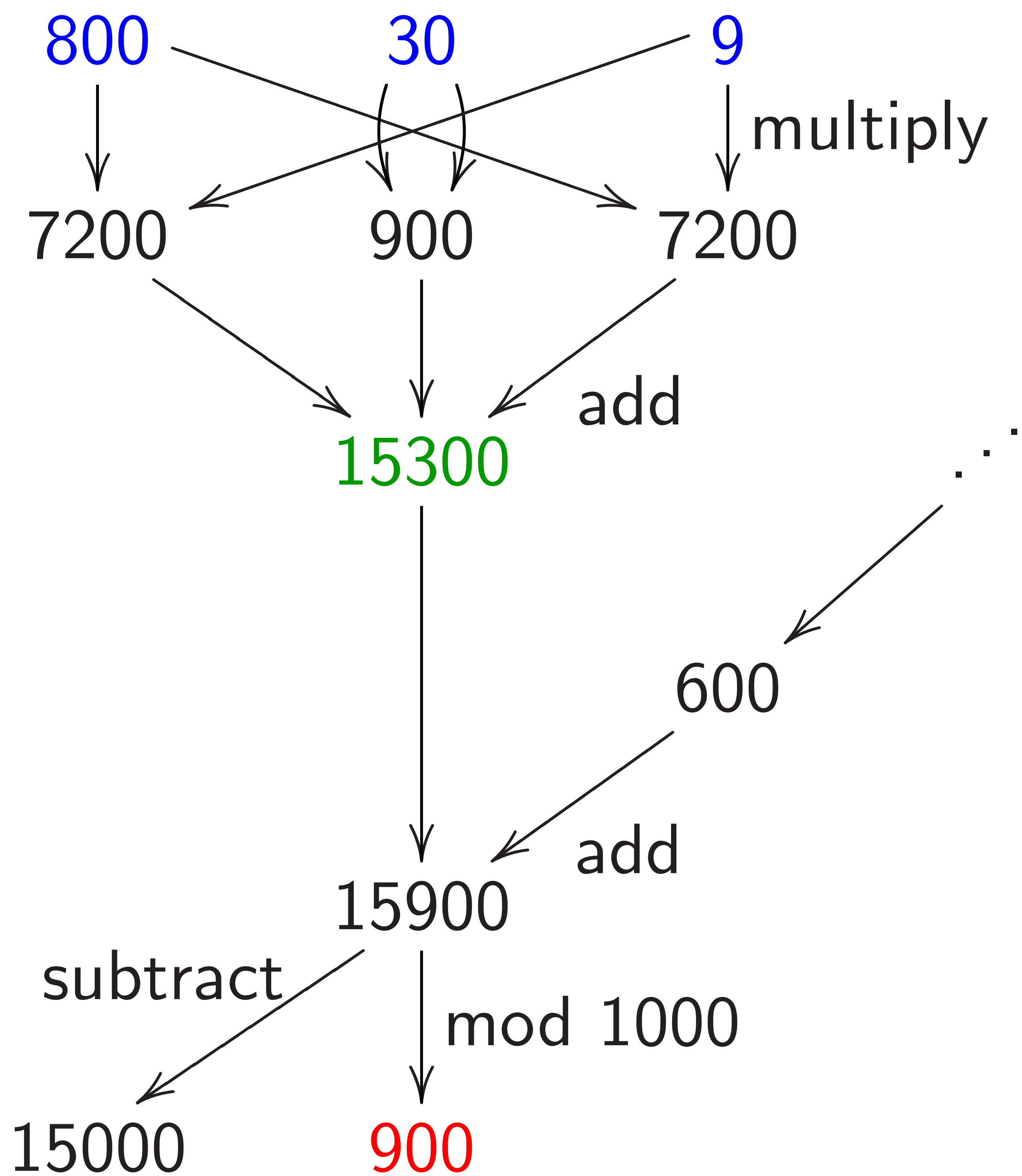
$t^3 + 15300t^2 +$

$t^3 + 15300t^2 +$

. .

$3000t^3+900t^2+$

What operations were used here?



Speedup: double i

$(\cdots + f_2 t^2 + f_1 t^1$

has coefficients su

$f_4 f_0 + f_3 f_1 + f_2 f_2 +$

5 mults, 4 adds.

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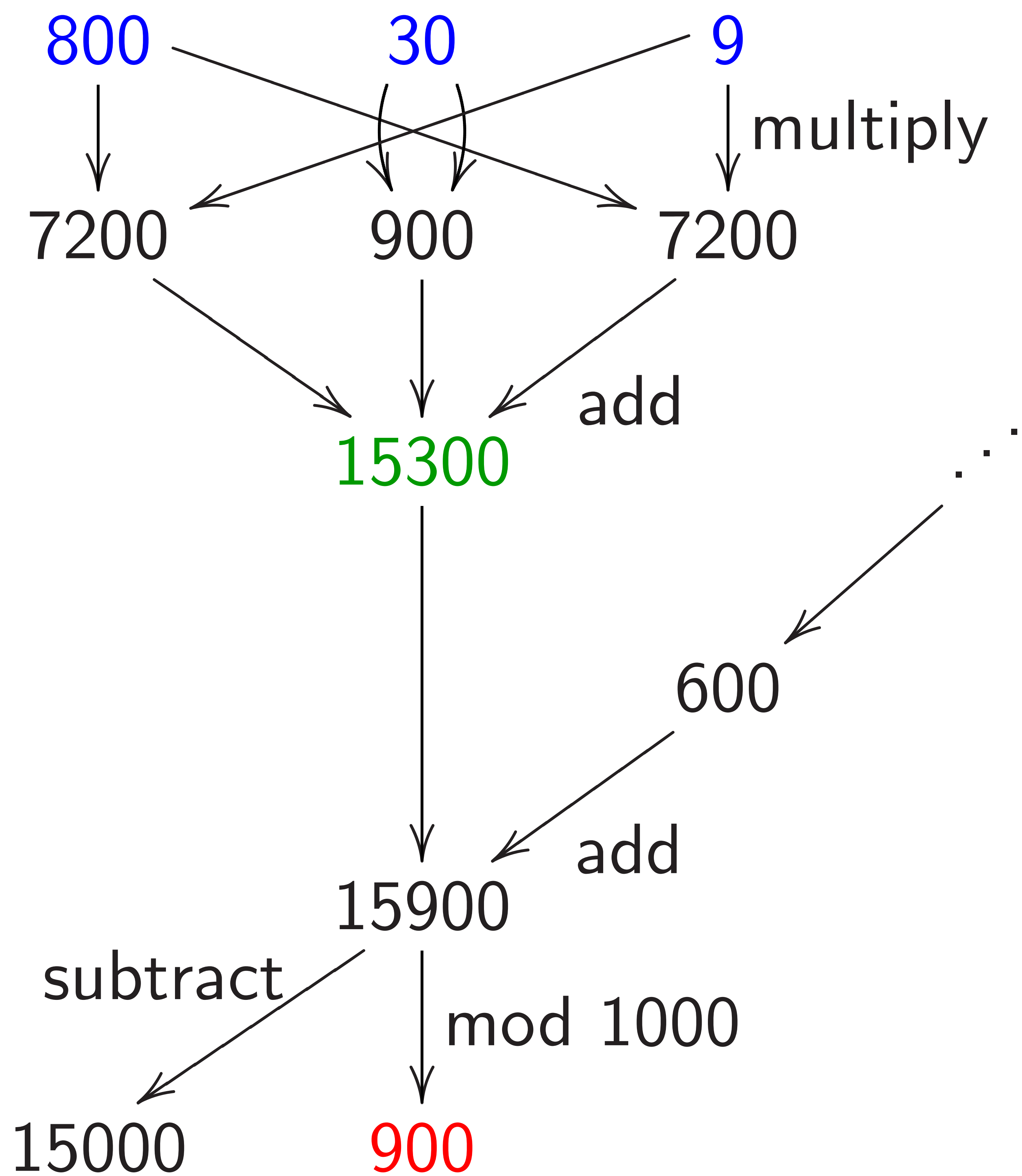
What operations were used here?



Speedup: double inside squa

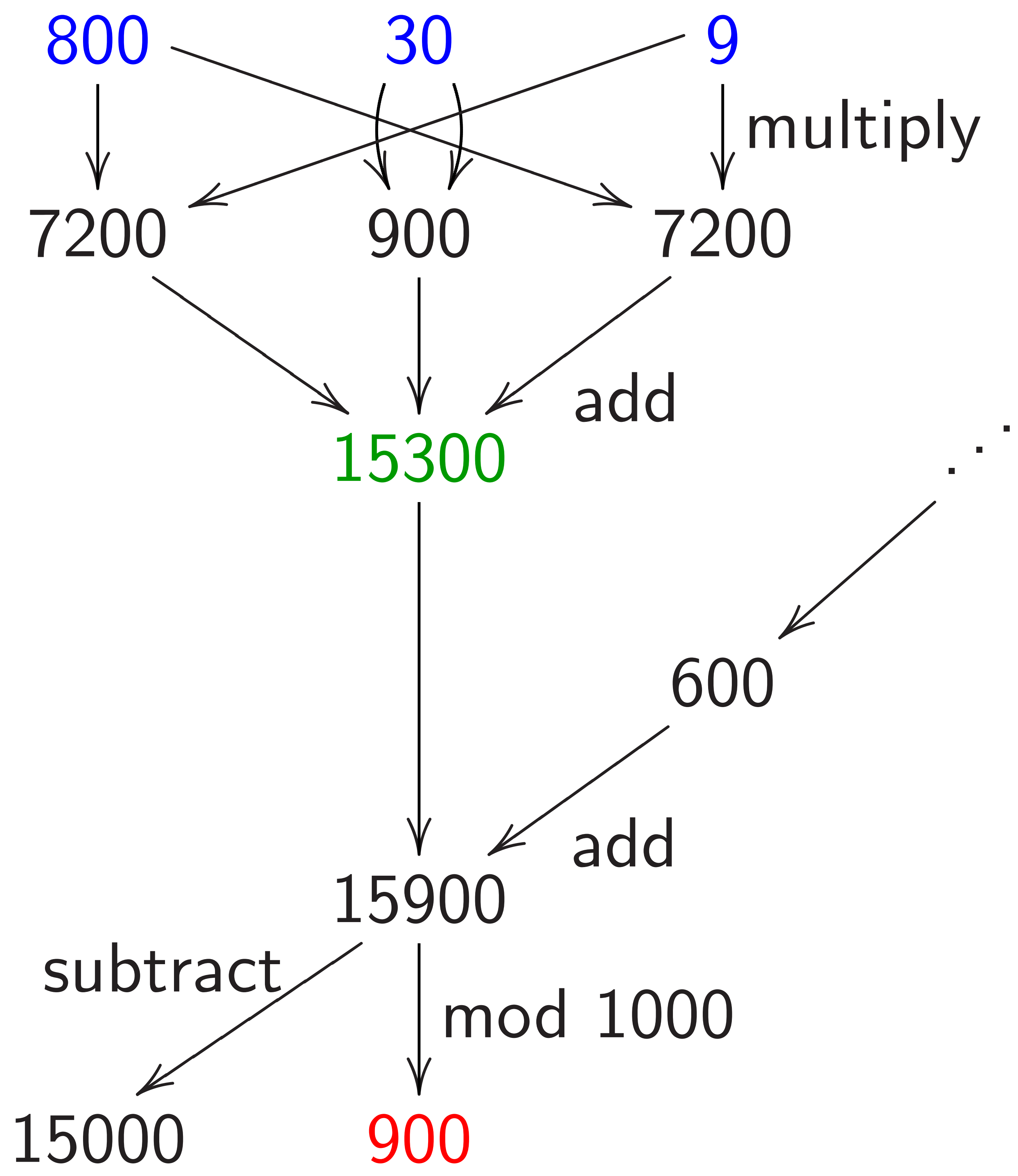
$(\cdots + f_2 t^2 + f_1 t^1 + f_0 t^0)^2$

has coefficients such as

$f_4 f_0 + f_3 f_1 + f_2 f_2 + f_1 f_3 + f_0$

5 mults, 4 adds.

What operations were used here?



Speedup: double inside squaring

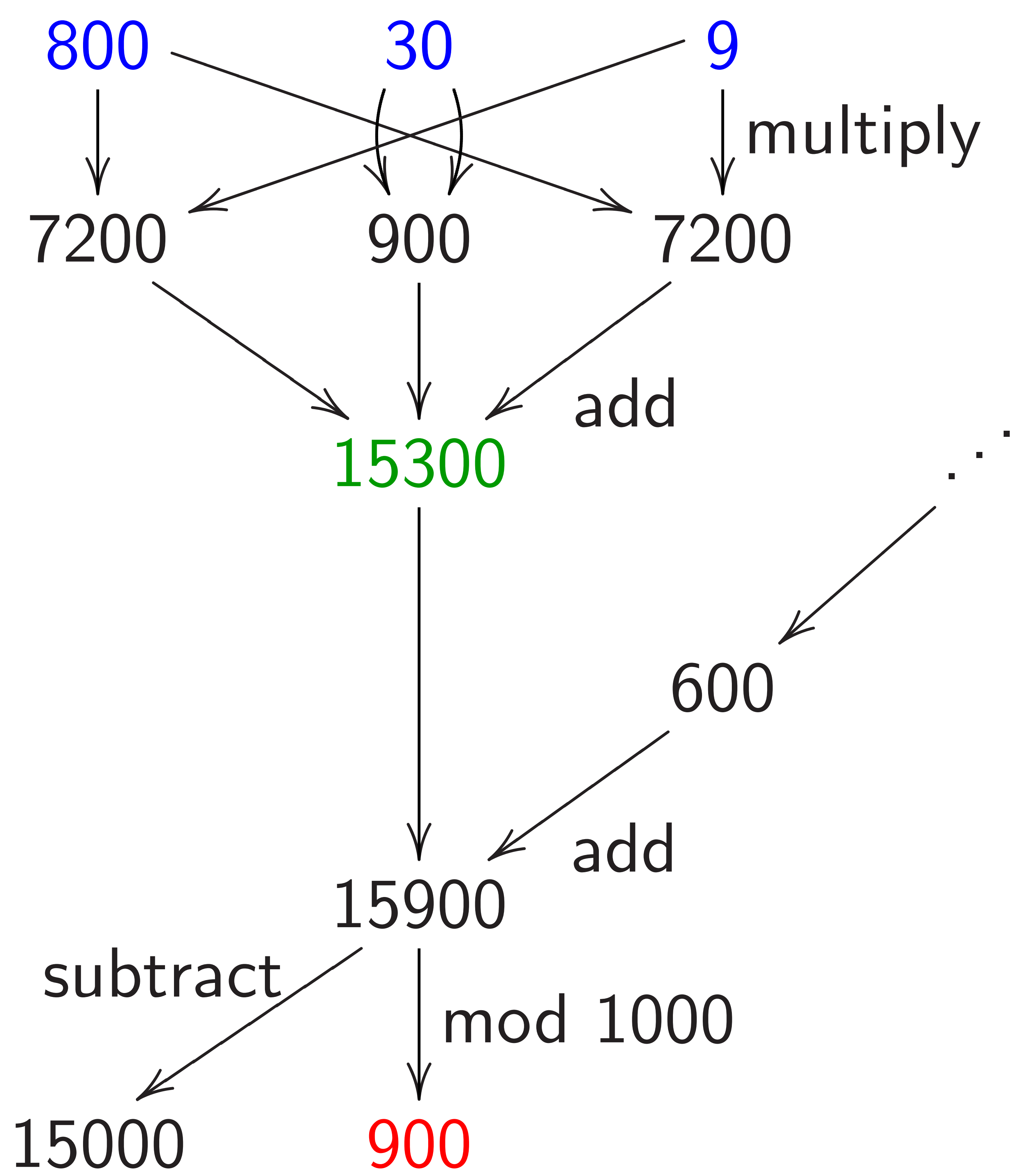
$(\cdots + f_2 t^2 + f_1 t^1 + f_0 t^0)^2$

has coefficients such as

$f_4 f_0 + f_3 f_1 + f_2 f_2 + f_1 f_3 + f_0 f_4$.

5 mults, 4 adds.

What operations were used here?



Speedup: double inside squaring

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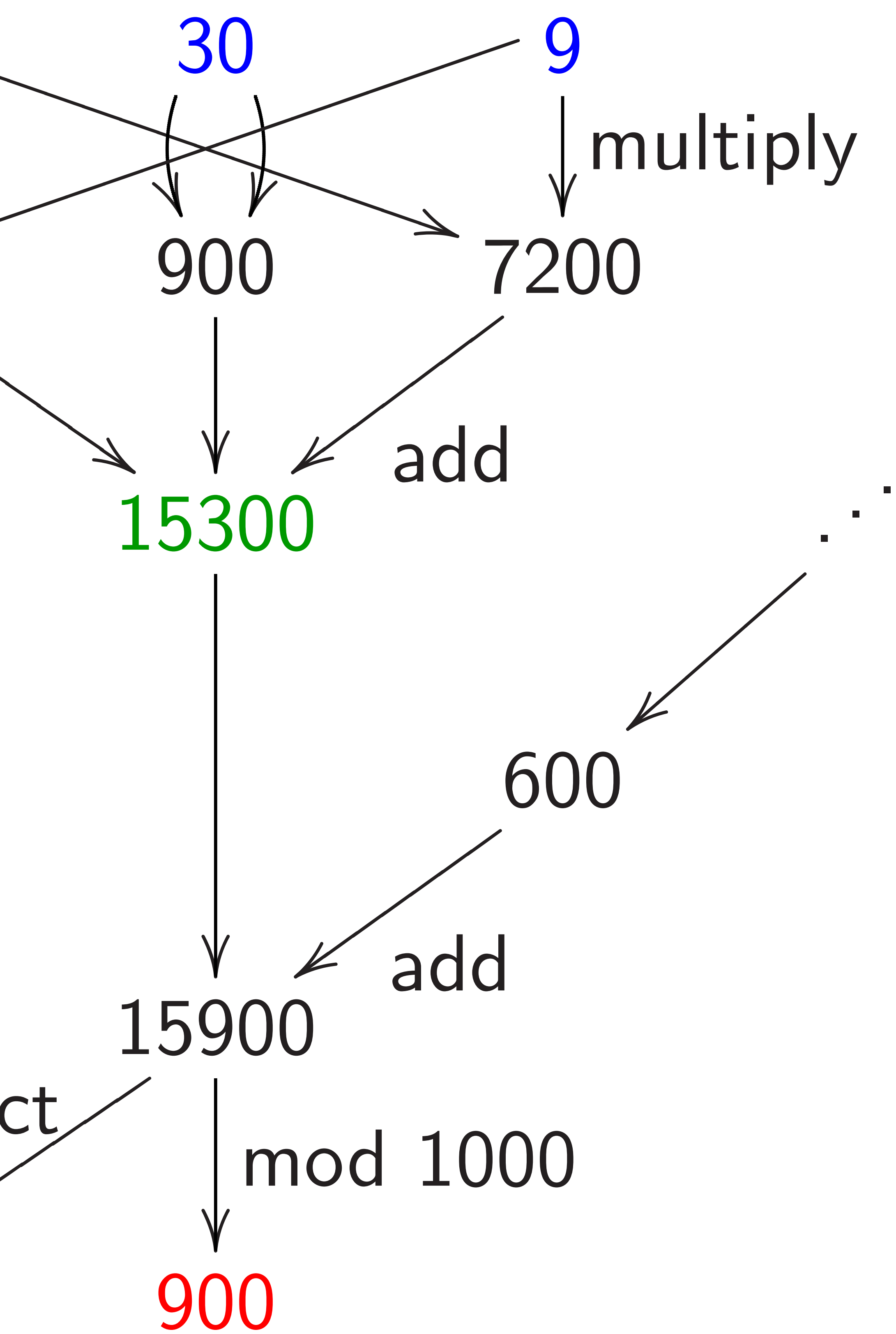
has coefficients such as
$f_4 f_0 + f_3 f_1 + f_2 f_2 + f_1 f_3 + f_0 f_4$.
5 mults, 4 adds.

Compute more efficiently as
$2 f_4 f_0 + 2 f_3 f_1 + f_2 f_2$.
3 mults, 2 adds, 2 doublings.

Save $\approx 1/2$ of the mults
if there are many coefficients.

...erations were used here?



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$2(f_4 f_0 + \ldots$
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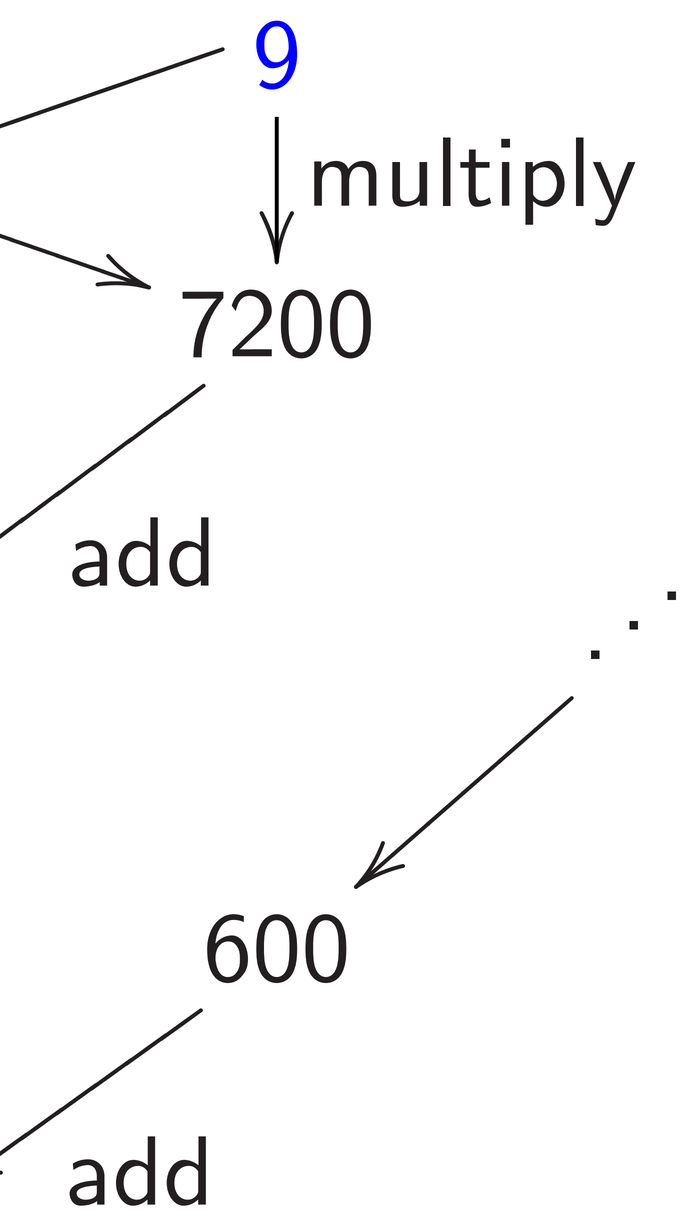

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Speedup: double inside squaring

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Faster alternative:
$2(f_4 f_0 + f_3 f_1) + f_2$
3 mults, 2 adds, 1

Save $\approx 1/2$ of the
if there are many

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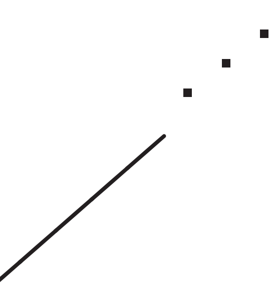

Speedup: double inside squaring

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3 mults, 2 adds, 1 doubling.

Save $\approx 1/2$ of the adds
if there are many coefficients.

Even faster alternative:
$(2 f_0) f_4 + (2 f_1) f_3 + f_2 f_2$,
after precomputing $2 f_0, 2 f_1, \ldots$.

3 mults, 2 adds, 0 doublings.
Precomputation ≈ 0.5 doublings.

p: double inside squaring

$_2 t^2 + f_1 t^1 + f_0 t^0)^2$

fficients such as

$f_1 + f_2 f_2 + f_1 f_3 + f_0 f_4$.

4 adds.

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Speedup: allow ne

Recall $159 \mapsto 15, 9$

Scaled: $15900 \mapsto$

Alternative: $159 \mapsto$

Scaled: $15900 \mapsto$

Use digits $\{-5, -4$

instead of $\{0, 1, \ldots$

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Speedup: allow negative coe

Recall $159 \mapsto 15, 9$.
Scaled: $15900 \mapsto 15000, 900$

Alternative: $159 \mapsto 16, -1$.
Scaled: $15900 \mapsto 16000, -1$

Use digits $\{-5, -4, \ldots, 4, 5\}$
instead of $\{0, 1, \ldots, 9\}$.
Small disadvantage: need $-$
Several small advantages:
easily handle negative intege
easily handle subtraction;
reduce products a bit.

Faster alternative:
$2(f_4 f_0 + f_3 f_1) + f_2 f_2$.
3 mults, 2 adds, 1 doubling.

Save $\approx 1/2$ of the adds
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Even faster alternative:
$(2f_0)f_4 + (2f_1)f_3 + f_2 f_2$,
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2 adds, 1 doubling.

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Speedup

Computi

multiply

square c

e.g. $a =$

$(3t^2+1t$

$6t^4 + 23$

carry: 8

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$64t^4 + 4$

$7t^5 + 0t$

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$7t^5 + 8t$

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$f_2 f_2$,
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0.5 doublings.

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Speedup: delay ca

Computing (e.g.)
multiply a, b polyn
square c poly, carr

e.g. $a = 314$, $b =$
$(3t^2+1t^1+4t^0)(2$
$6t^4 + 23t^3 + 18t^2$
carry: $8t^4 + 5t^3 +$

As before $(8t^2 + 3$
$64t^4 + 48t^3 + 153t$
$7t^5 + 0t^4 + 3t^3 +$

$+$: $7t^5+8t^4+8t^3$
$7t^5 + 8t^4 + 9t^3 +$

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Speedup: delay carries

Computing (e.g.) big $ab + c$

multiply a, b polynomials, ca

square c poly, carry, add, ca

e.g. $a = 314$, $b = 271$, $c =$

$(3t^2+1t^1+4t^0)(2t^2+7t^1+1$

$6t^4 + 23t^3 + 18t^2 + 29t^1 +$

carry: $8t^4 + 5t^3 + 0t^2 + 9t^1$

As before $(8t^2 + 3t^1 + 9t^0)$

$64t^4 + 48t^3 + 153t^2 + 54t^1$

$7t^5 + 0t^4 + 3t^3 + 9t^2 + 2t^1$

$+$: $7t^5+8t^4+8t^3+9t^2+11t$

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Use digits $\{-5, -4, \ldots, 4, 5\}$
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Small disadvantage: need $-$.
Several small advantages:
easily handle negative integers;
easily handle subtraction;
reduce products a bit.

Speedup: delay carries

Computing (e.g.) big $ab + c^2$:
multiply a, b polynomials, carry,
square c poly, carry, add, carry.

e.g. $a = 314$, $b = 271$, $c = 839$:
$(3t^2+1t^1+4t^0)(2t^2+7t^1+1t^0) =$
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Speedup: delay carries

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Faster: multiply a

square c polynomi

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Faster: multiply a, b polynon
square c polynomial, add, ca

$(6t^4 + 23t^3 + 18t^2 + 29t^1 + 4$
$(64t^4 + 48t^3 + 153t^2 + 54t^1 +$
$= 70t^4 + 71t^3 + 171t^2 + 83t^1 +$
$7t^5 + 8t^4 + 9t^3 + 0t^2 + 1t^1$

Eliminate intermediate carrie
Outweighs cost of handling
slightly larger coefficients.

Important to carry between
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but carries are usually a bad
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Speedup: delay carries

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$(6t^4 + 23t^3 + 18t^2 + 29t^1 + 4t^0) +$
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o: delay carries

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a, b polynomials, carry,

r poly, carry, add, carry.

= 314, $b = 271$, $c = 839$:

$^1 + 4t^0)(2t^2 + 7t^1 + 1t^0) =$

$3t^3 + 18t^2 + 29t^1 + 4t^0$;

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re $(8t^2 + 3t^1 + 9t^0)^2 =$

$8t^3 + 153t^2 + 54t^1 + 81t^0$;

$^4 + 3t^3 + 9t^2 + 2t^1 + 1t^0$.

$8t^4 + 8t^3 + 9t^2 + 11t^1 + 5t^0$;

$^4 + 9t^3 + 0t^2 + 1t^1 + 5t^0$.

Faster: multiply a, b polynomials, square c polynomial, add, carry.

$(6t^4 + 23t^3 + 18t^2 + 29t^1 + 4t^0) +$
$(64t^4 + 48t^3 + 153t^2 + 54t^1 + 81t^0)$
$= 70t^4 + 71t^3 + 171t^2 + 83t^1 + 85t^0$;
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Speedup

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$f = f_0 +$

$g = g_0$

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Faster:

$F_0 = f_0$

$F_1 = f_{10}$

Similarly

Then $f g$

$+ (F_0 G_0$

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big $ab + c^2$:

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ry, add, carry.

271, $c = 839$:

$t^2 + 7t^1 + 1t^0) =$

$+ 29t^1 + 4t^0;$

$0t^2 + 9t^1 + 4t^0.$

$3t^1 + 9t^0)^2 =$

$t^2 + 54t^1 + 81t^0;$

$9t^2 + 2t^1 + 1t^0.$

$+9t^2 + 11t^1 + 5t^0;$

$0t^2 + 1t^1 + 5t^0.$

Faster: multiply a, b polynomials, square c polynomial, add, carry.

$$(6t^4 + 23t^3 + 18t^2 + 29t^1 + 4t^0) + (64t^4 + 48t^3 + 153t^2 + 54t^1 + 81t^0) = 70t^4 + 71t^3 + 171t^2 + 83t^1 + 85t^0;$$

$$7t^5 + 8t^4 + 9t^3 + 0t^2 + 1t^1 + 5t^0.$$

Eliminate intermediate carries. Outweighs cost of handling slightly larger coefficients.

Important to carry between multiplications (and squarings) to reduce coefficient size; but carries are usually a bad idea before additions, subtractions, etc.

Speedup: polynom

How much work t

$f = f_0 + f_1 t + \cdots$

$g = g_0 + g_1 t + \cdots$

Using the obvious

400 coeff mults, 3

Faster: Write f as

$F_0 = f_0 + f_1 t + \cdots$

$F_1 = f_{10} + f_{11} t +$

Similarly write g a

Then $fg = (F_0 +$

$+ (F_0 G_0 - F_1 G_1 t^1$

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839:

$t^0) =$

$4t^0$;

$+ 4t^0$.

$^2 =$

$+81t^0$;

$+ 1t^0$.

$^1 + 5t^0$;

$+ 5t^0$.

Faster: multiply a, b polynomials, square c polynomial, add, carry.

$(6t^4 + 23t^3 + 18t^2 + 29t^1 + 4t^0) +$
$(64t^4 + 48t^3 + 153t^2 + 54t^1 + 81t^0)$
$= 70t^4 + 71t^3 + 171t^2 + 83t^1 + 85t^0$;
$7t^5 + 8t^4 + 9t^3 + 0t^2 + 1t^1 + 5t^0$.

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Speedup: polynomial Karats

How much work to multiply
$f = f_0 + f_1 t + \cdots + f_{19} t^{19}$,
$g = g_0 + g_1 t + \cdots + g_{19} t^{19}$

Using the obvious method:
400 coeff mults, 361 coeff a

Faster: Write f as $F_0 + F_1 t$
$F_0 = f_0 + f_1 t + \cdots + f_9 t^9$;
$F_1 = f_{10} + f_{11} t + \cdots + f_{19} t^9$
Similarly write g as $G_0 + G_1$

Then $fg = (F_0 + F_1)(G_0 +$
$+ (F_0 G_0 - F_1 G_1 t^{10})(1 - t^{10}$

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Speedup: polynomial Karatsuba

How much work to multiply polys
$f = f_0 + f_1 t + \cdots + f_{19} t^{19}$,
$g = g_0 + g_1 t + \cdots + g_{19} t^{19}$?

Using the obvious method: 400 coeff mults, 361 coeff adds.

Faster: Write f as $F_0 + F_1 t^{10}$;
$F_0 = f_0 + f_1 t + \cdots + f_9 t^9$;
$F_1 = f_{10} + f_{11} t + \cdots + f_{19} t^9$.
Similarly write g as $G_0 + G_1 t^{10}$.

Then $fg = (F_0 + F_1)(G_0 + G_1) t^{10}$
$+ (F_0 G_0 - F_1 G_1 t^{10})(1 - t^{10})$.

multiply a, b polynomials,
polynomial, add, carry.

$3t^3 + 18t^2 + 29t^1 + 4t^0) +$
$48t^3 + 153t^2 + 54t^1 + 81t^0)$
$71t^3 + 171t^2 + 83t^1 + 85t^0;$
$^4 + 9t^3 + 0t^2 + 1t^1 + 5t^0.$

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$+ (F_0 G_0 - F_1 G_1 t^{10})(1 - t^{10})$.

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20 adds for $F_0 + F$
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Total 300 mults, 3
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20 adds for $F_0 + F_1$, $G_0 + G$
300 mults for three products
$F_0 G_0$, $F_1 G_1$, $(F_0 + F_1)(G_0 +$
243 adds for those products
9 adds for $F_0 G_0 - F_1 G_1 t^{10}$
with subs counted as adds
and with delayed negations.
19 adds for $\cdots (1 - t^{10})$.
19 adds to finish.

Total 300 mults, 310 adds.
Larger coefficients, slight ex
still saves time.

Can apply idea recursively
as poly degree grows.

Speedup: polynomial Karatsuba

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ne obvious method:

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Write f as $F_0 + F_1 t^{10}$;

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$\cdots + f_{19} t^9$.
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$F_1)(G_0 + G_1)t^{10}$
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20 adds for $F_0 + F_1$, $G_0 + G_1$.
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Many other algebraic speedu
in polynomial multiplication
"Toom," "FFT," etc.

Increasingly important as
polynomial degree grows.
$O(n \lg n \lg \lg n)$ coeff operati
to compute n-coeff product.

Useful for sizes of n
that occur in cryptography?
In some cases, yes!
But Karatsuba is the limit
for prime-field ECC/ECDLP
on most current CPUs.

20 adds for $F_0 + F_1$, $G_0 + G_1$.
300 mults for three products
F_0G_0, F_1G_1, $(F_0 + F_1)(G_0 + G_1)$.
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Modular reduction

How to compute f

Can use definition
$f \bmod p = f - p \lfloor$
Can multiply f by
precomputed $1/p$
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Slight speedup: "2
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Modular reduction

How to compute $f \bmod p$?

Can use definition: $f \bmod p = f - p \lfloor f/p \rfloor$. Can multiply f by a precomputed $1/p$ approxima easily adjust to obtain $\lfloor f/p$

Slight speedup: "2-adic inve "Montgomery reduction."

Many other algebraic speedups in polynomial multiplication: "Toom," "FFT," etc.

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Slight speedup: "2-adic inverse";
"Montgomery reduction."

e.g. 314

Precomp
$\lfloor 100000$
$= 36787$

Comput
314159
$= 11557$

Comput
3141592
$= 57823$
Oops, to
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Modular reduction

How to compute f mod p?

Can use definition:
$f \bmod p = f - p\lfloor f/p \rfloor$.
Can multiply f by a
precomputed $1/p$ approximation;
easily adjust to obtain $\lfloor f/p \rfloor$.

Slight speedup: "2-adic inverse";
"Montgomery reduction."

e.g. 31415926535

Precompute
$\lfloor 1000000000000/$
$= 3678796$.

Compute
$314159 \cdot 3678796$
$= 1155726872564$

Compute
$31415926535 8 - 1$
$= 578230$.
Oops, too big:
$578230 - 271828$
$306402 - 271828$

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Modular reduction

How to compute f mod p?

Can use definition:
$f \bmod p = f - p \lfloor f/p \rfloor$.
Can multiply f by a precomputed $1/p$ approximation; easily adjust to obtain $\lfloor f/p \rfloor$.

Slight speedup: "2-adic inverse"; "Montgomery reduction."

e.g. 314159265358 mod 271

Precompute
$\lfloor 1000000000000/271828 \rfloor$
$= 3678796$.

Compute
$314159 \cdot 3678796$
$= 1155726872564$.

Compute
$314159265358 - 1155726 \cdot 2$
$= 578230$.
Oops, too big:
$578230 - 271828 = 306402$.
$306402 - 271828 = 34574$.

Modular reduction

How to compute $f \bmod p$?

Can use definition:
$f \bmod p = f - p \lfloor f/p \rfloor$.
Can multiply f by a precomputed $1/p$ approximation; easily adjust to obtain $\lfloor f/p \rfloor$.

Slight speedup: "2-adic inverse"; "Montgomery reduction."

e.g. 314159265358 mod 271828:

Precompute
$\lfloor 1000000000000/271828 \rfloor$
$= 3678796$.

Compute
$314159 \cdot 3678796$
$= 1155726872564$.

Compute
$314159265358 - 1155726 \cdot 271828$
$= 578230$.
Oops, too big:
$578230 - 271828 = 306402$.
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compute f mod p?

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e.g. 314159265358 mod 271828:

Precompute

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$= 578230$.

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Compute

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Compute

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We can do better:

p is chosen with a

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Special primes hur

for $\mathbf{F}_p^*$, Clock($\mathbf{F}_p$),

but not for elliptic

Curve25519: $p =$

NIST P-224: $p =$

secp112r1: $p = (2$

Divides special for

gls1271: $p = 2^{127}$

degree-2 extension

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e.g. 314159265358 mod 271828:

Precompute

$\lfloor 1000000000000/271828 \rfloor$

$= 3678796$.

Compute

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$= 578230$.

Oops, too big:

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$306402 - 271828 = 34574$.

We can do better: normally

p is chosen with a special fo

to make f mod p much fast

Special primes hurt security

for $\mathbf{F}_p^*$, Clock($\mathbf{F}_p$), etc.,

but not for elliptic curves!

Curve25519: $p = 2^{255} - 19$.

NIST P-224: $p = 2^{224} - 2^{9}$

secp112r1: $p = (2^{128} - 3)/$

Divides special form.

gls1271: $p = 2^{127} - 1$, with

degree-2 extension (a bit sca

e.g. 314159265358 mod 271828:

Precompute

$\lfloor 1000000000000/271828 \rfloor$

$= 3678796$.

Compute

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Compute

$314159265358 - 1155726 \cdot 271828$

$= 578230$.

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$578230 - 271828 = 306402$.

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We can do better: normally p is chosen with a special form to make f mod p much faster.

Special primes hurt security for $\mathbf{F}_p^*$, Clock($\mathbf{F}_p$), etc., but not for elliptic curves!

Curve25519: $p = 2^{255} - 19$.

NIST P-224: $p = 2^{224} - 2^{96} + 1$.

secp112r1: $p = (2^{128} - 3)/76439$.
Divides special form.

gls1271: $p = 2^{127} - 1$, with degree-2 extension (a bit scary).

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We can do better: normally p is chosen with a special form to make f mod p much faster.

Special primes hurt security for $\mathbf{F}_p^*$, Clock($\mathbf{F}_p$), etc., but not for elliptic curves!

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gls1271: $p = 2^{127} - 1$, with degree-2 extension (a bit scary).

Small example: p

Then $1000000a +$

e.g. 31415926535

$314159 \cdot 1000000$

$314159(-3) + 265$

$-942477 + 26535$

-677119.

Easily adjust $b - 3$

to the range $\{0, 1,$

by adding/subtrac

e.g. $-677119 \equiv 3$

.828:

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We can do better: normally p is chosen with a special form to make f mod p much faster.

Special primes hurt security for $\mathbf{F}_p^*$, Clock($\mathbf{F}_p$), etc., but not for elliptic curves!

Curve25519: $p = 2^{255} - 19$.

NIST P-224: $p = 2^{224} - 2^{96} + 1$.

secp112r1: $p = (2^{128} - 3)/76439$. *Divides* special form.

gls1271: $p = 2^{127} - 1$, with degree-2 extension (a bit scary).

Small example: $p = 100000$

Then $1000000a + b \equiv b - 3$

e.g. $314159265358 =$
$314159 \cdot 1000000 + 265358$
$314159(-3) + 265358 =$
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-677119.

Easily adjust $b - 3a$ to the range $\{0, 1, \ldots, p - 1$ by adding/subtracting a few e.g. $-677119 \equiv 322884$.

We can do better: normally p is chosen with a special form to make f mod p much faster.

Special primes hurt security for $\mathbf{F}_p^*$, Clock($\mathbf{F}_p$), etc., but not for elliptic curves!

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in $\mathbf{Z}/10$

$3t^5 + 1t$

obtainin

$14t^7 + 4$

$82t^3 + 4$

Reduce:

$(-3c_i)t^i$

$64t^3 - 3$

Carry: 8

$1t^3 + 2t$

$= 1000003$.

$b \equiv b - 3a$.

$3 =$

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Can delay carries u

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e.g. To square 314

in $\mathbf{Z}/1000003$: Sq

$3t^5 + 1t^4 + 4t^3 +$

obtaining $9t^{10} + 6$

$14t^7 + 48t^6 + 72t$

$82t^3 + 43t^2 + 90t$

Reduce: replace (c

$(-3c_i)t^i$, obtainin

$64t^3 - 32t^2 + 48t$

Carry: $8t^6 - 4t^5 -$

$1t^3 + 2t^2 + 2t^1 -$

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Can delay carries until after multiplication by 3.

e.g. To square 314159 in $\mathbf{Z}/1000003$: Square poly $3t^5 + 1t^4 + 4t^3 + 1t^2 + 5t^1$ obtaining $9t^{10} + 6t^9 + 25t^8$ $14t^7 + 48t^6 + 72t^5 + 59t^4$ $82t^3 + 43t^2 + 90t^1 + 81t^0$.

Reduce: replace $(c_i)t^{6+i}$ by $(-3c_i)t^i$, obtaining $72t^5 +$ $64t^3 - 32t^2 + 48t^1 - 63t^0$.

Carry: $8t^6 - 4t^5 - 2t^4 +$ $1t^3 + 2t^2 + 2t^1 - 3t^0$.

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To minin
mix redu
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e.g. Star
$25t^8 + 1$
$82t^3 + 4$

Reduce
$t^5 \rightarrow t^6$
$56t^6 - 5$
$90t^1 + 8$

Finish re
$64t^3 - 3$
$t^0 \rightarrow t^1$
$-4t^5 - 2$

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To minimize poly
mix reduction and
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e.g. Start from squ
$25t^8 + 14t^7 + 48t^6$
$82t^3 + 43t^2 + 90t$

Reduce $t^{10} \to t^4$
$t^5 \to t^6$: $6t^9 + 2$
$56t^6 - 5t^5 + 2t^4$
$90t^1 + 81t^0$.

Finish reduction:
$64t^3 - 32t^2 + 48t$
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Reduce $t^{10} \to t^4$ and carry
$t^5 \to t^6$: $6t^9 + 25t^8 + 14t$
$56t^6 - 5t^5 + 2t^4 + 82t^3 + 4$
$90t^1 + 81t^0$.

Finish reduction: $-5t^5 + 2t$
$64t^3 - 32t^2 + 48t^1 - 87t^0$.
$t^0 \to t^1 \to t^2 \to t^3 \to t^4$ –
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$^4 + 4t^3 + 1t^2 + 5t^1 + 9t^0$,
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Speedup

$p = 2^{61}$

Five coe
$f_4 t^4 + f_3$
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Reduce:
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<u>Speedup: non-inte</u>

$p = 2^{61} - 1$.

Five coeffs in radi
$f_4 t^4 + f_3 t^3 + f_2 t^2$
Most coeffs could

Square $\cdots + 2(f_4 f$
Coeff of t^5 could

Reduce: $2^{65} = 2^4$
$\cdots + (2^5(f_4 f_1 + f_3$
Coeff could be $>$
Very little room fo
additions, delayed
on 32-bit platform

$+ 9t^0$,

$+$

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$32t^4 +$

To minimize poly degree, mix reduction and carrying, carrying the top sooner.

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Speedup: non-integer radix

$p = 2^{61} - 1$.

Five coeffs in radix 2^{13}? $f_4 t^4 + f_3 t^3 + f_2 t^2 + f_1 t^1 + f$
Most coeffs could be 2^{12}.

Square $\cdots + 2(f_4 f_1 + f_3 f_2)t^5$
Coeff of t^5 could be $> 2^{25}$.

Reduce: $2^{65} = 2^4$ in $\mathbf{Z}/(2^{61}$
$\cdots + (2^5(f_4 f_1 + f_3 f_2) + f_0^2)t$
Coeff could be $> 2^{29}$.
Very little room for additions, delayed carries, et on 32-bit platforms.

To minimize poly degree,
mix reduction and carrying,
carrying the top sooner.

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Reduce $t^{10}\to t^4$ and carry $t^4\to t^5\to t^6$: $6t^9+25t^8+14t^7+56t^6-5t^5+2t^4+82t^3+43t^2+90t^1+81t^0$.

Finish reduction: $-5t^5+2t^4+64t^3-32t^2+48t^1-87t^0$. Carry $t^0\to t^1\to t^2\to t^3\to t^4\to t^5$: $-4t^5-2t^4+1t^3+2t^2-1t^1+3t^0$.

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$\to t^2 \to t^3 \to t^4 \to t^5$:
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Coeff could be $> 2^{29}$.
Very little room for
additions, delayed carries, etc.
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Scaled:
f_4 is mu
f_3 is mu
f_2 is mu
f_1 is mu
f_0 is mu
$\cdots + (2$

Better:
f_4 is mu
f_3 is mu
f_2 is mu
f_1 is mu
f_0 is mu
Saves a

degree,
carrying,
ooner.

uare $9t^{10}+6t^9+$
$+72t^5+59t^4+$
$^1+81t^0$.

and carry $t^4 \to$
$5t^8+14t^7+$
$+82t^3+43t^2+$

$-5t^5+2t^4+$
$^1-87t^0$. Carry
$t^3 \to t^4 \to t^5$:
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Coeff could be $> 2^{29}$.
Very little room for
additions, delayed carries, etc.
on 32-bit platforms.

Scaled: Evaluate a
f_4 is multiple of 2^5
f_3 is multiple of 2^3
f_2 is multiple of 2^2
f_1 is multiple of 2^1
f_0 is multiple of 2^0
$\cdots + (2^{-60}(f_4 f_1 +$

Better: Non-intege
f_4 is multiple of 2^4
f_3 is multiple of 2^3
f_2 is multiple of 2^2
f_1 is multiple of 2^1
f_0 is multiple of 2^0
Saves a few bits in

$+6t^9+$

$59t^4+$

$t^4 \rightarrow$

$7 +$

$43t^2+$

$4 +$

Carry

$\rightarrow t^5$:

$+3t^0$.

Speedup: non-integer radix

$p = 2^{61} - 1$.

Five coeffs in radix 2^{13}?
$f_4 t^4 + f_3 t^3 + f_2 t^2 + f_1 t^1 + f_0 t^0$.
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Reduce: $2^{65} = 2^4$ in $\mathbf{Z}/(2^{61} - 1)$;
$\cdots + (2^5(f_4 f_1 + f_3 f_2) + f_0^2)t^0$.
Coeff could be $> 2^{29}$.
Very little room for
additions, delayed carries, etc.
on 32-bit platforms.

Scaled: Evaluate at $t = 1$.
f_4 is multiple of 2^{52};
f_3 is multiple of 2^{39};
f_2 is multiple of 2^{26};
f_1 is multiple of 2^{13};
f_0 is multiple of 2^0. Reduce:
$\cdots + (2^{-60}(f_4 f_1 + f_3 f_2) + f_0^2$

Better: Non-integer radix 2
f_4 is multiple of 2^{49};
f_3 is multiple of 2^{37};
f_2 is multiple of 2^{25};
f_1 is multiple of 2^{13};
f_0 is multiple of 2^0.
Saves a few bits in coeffs.

Speedup: non-integer radix

$p = 2^{61} - 1$.

Five coeffs in radix 2^{13}?
$f_4 t^4 + f_3 t^3 + f_2 t^2 + f_1 t^1 + f_0 t^0$.
Most coeffs could be 2^{12}.

Square $\cdots + 2(f_4 f_1 + f_3 f_2) t^5 + \cdots$.
Coeff of t^5 could be $> 2^{25}$.

Reduce: $2^{65} = 2^4$ in $\mathbf{Z}/(2^{61} - 1)$;
$\cdots + (2^5 (f_4 f_1 + f_3 f_2) + f_0^2) t^0$.
Coeff could be $> 2^{29}$.
Very little room for
additions, delayed carries, etc.
on 32-bit platforms.

Scaled: Evaluate at $t = 1$.
f_4 is multiple of 2^{52};
f_3 is multiple of 2^{39};
f_2 is multiple of 2^{26};
f_1 is multiple of 2^{13};
f_0 is multiple of 2^0. Reduce:
$\cdots + (2^{-60}(f_4 f_1 + f_3 f_2) + f_0^2) t^0$.

Better: Non-integer radix $2^{12.2}$.
f_4 is multiple of 2^{49};
f_3 is multiple of 2^{37};
f_2 is multiple of 2^{25};
f_1 is multiple of 2^{13};
f_0 is multiple of 2^0.
Saves a few bits in coeffs.

: non-integer radix

$- 1$.

ffs in radix 2^{13}?

$_3 t^3 + f_2 t^2 + f_1 t^1 + f_0 t^0$.

effs could be 2^{12}.

$\cdots + 2(f_4 f_1 + f_3 f_2) t^5 + \cdots$.

t^5 could be $> 2^{25}$.

$2^{65} = 2^4$ in $\mathbf{Z}/(2^{61} - 1)$;

$^5(f_4 f_1 + f_3 f_2) + f_0^2) t^0$.

uld be $> 2^{29}$.

le room for

s, delayed carries, etc.

t platforms.

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Saves a few bits in coeffs.

More ba

NIST P-

$2^{256} - 2$

i.e. $t^8 -$

evaluate

eger radix

× 2^{13}?

$+ f_1 t^1 + f_0 t^0$.

be 2^{12}.

$_1 + f_3 f_2) t^5 + \cdots$.

be $> 2^{25}$.

in $\mathbf{Z}/(2^{61} - 1)$;

$f_2) + f_0^2) t^0$.

2^{29}.

or

carries, etc.

s.

Scaled: Evaluate at $t = 1$.

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More bad choices

NIST P-256 prime
$2^{256} - 2^{224} + 2^{192}$
i.e. $t^8 - t^7 + t^6 +$
evaluated at $t = 2$

$f_0 t^0$.

$+ \cdots$.

$- 1)$;

0.

c.

Scaled: Evaluate at $t = 1$.

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Saves a few bits in coeffs.

More bad choices from NIST

NIST P-256 prime:

$2^{256} - 2^{224} + 2^{192} + 2^{96} - 1$

i.e. $t^8 - t^7 + t^6 + t^3 - 1$

evaluated at $t = 2^{32}$.

Scaled: Evaluate at $t = 1$.

f_4 is multiple of 2^{52};

f_3 is multiple of 2^{39};

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i.e. $t^8 - t^7 + t^6 + t^3 - 1$

evaluated at $t = 2^{32}$.

Reduction: replace $c_i t^{8+i}$ with

$c_i t^{7+i} - c_i t^{6+i} - c_i t^{3+i} + c_i t^i$.

Minor problem: often slower than

small const mult and one add.

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evaluated at $t = 2^{32}$.

Reduction: replace $c_i t^{8+i}$ with

$c_i t^{7+i} - c_i t^{6+i} - c_i t^{3+i} + c_i t^i$.

Minor problem: often slower than

small const mult and one add.

Major problem: With radix 2^{32},

products are almost 2^{64}.

Sums are slightly above 2^{64}:

bad for every common CPU.

Need very frequent carries.