Computational algebraic number theory tackles lattice-based cryptography

Daniel J. Bernstein
University of Illinois at Chicago &
Technische Universiteit Eindhoven

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Big generator
Moving through the night
—Yes, "Big Generator", 1987

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"I think NTRU should switch to random prime-degree extensions with big Galois groups."

2014.02 blog post:
"Here's a concrete suggestion, which I'll call NTRU Prime, for eliminating the structures that I find worrisome in existing ideal-lattice-based encryption systems."

NTRU Prime uses primes p, q with field $(\mathbf{Z}/q)[x]/(x^p - x - 1)$.

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Nonzero ideals of $\mathcal{O}$ factor uniquely as products of powers of prime ideals of $\mathcal{O}$.

The short-generator problem

Take degree-n number field K.
i.e. field $K \subseteq \mathbf{C}$ with $\operatorname{len}_{\mathbf{Q}} K = n$.

(Weaker specification: field K with $\mathbf{Q} \subseteq K$ and $\operatorname{len}_{\mathbf{Q}} K = n$.)

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The short-generat

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e.g. $\zeta = \exp(\pi i/4$

$\mathcal{O} = \mathbf{Z}[\zeta] \hookrightarrow\!\!\!\!\twoheadrightarrow \mathbf{Z}[x]/$

The $\mathbf{Z}$-submodule

$201 - 233\zeta - 430$

$935 - 1063\zeta - 198$

$979 - 1119\zeta - 209$

$718 - 829\zeta - 153$

is an ideal I of $\mathcal{O}$.

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Find "short" nonzero $g \in \mathcal{O}$

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The $\mathbf{Z}$-submodule of $\mathcal{O}$ gen

$201 - 233\zeta - 430\zeta^2 - 712\zeta$

$935 - 1063\zeta - 1986\zeta^2 - 32$

$979 - 1119\zeta - 2092\zeta^2 - 34$

$718 - 829\zeta - 1537\zeta^2 - 254$

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<u>The latt</u>

Use LLL
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$\mathbf{Z}A + \mathbf{Z}$
$A = (20$
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$\mathbf{Q}[x]/(x^2+1)$
$[x]/(x^2+1)$.
56), $K=\mathbf{Q}(\zeta)$
$[x]/(x^{256}+1)$.
661), $K=\mathbf{Q}(\zeta)$
$\ddots$
$\Rightarrow \mathcal{O}=$
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The $\mathbf{Z}$-submodule of $\mathcal{O}$ gen by
$201-233\zeta-430\zeta^2-712\zeta^3$,
$935-1063\zeta-1986\zeta^2-3299\zeta^3$,
$979-1119\zeta-2092\zeta^2-3470\zeta^3$,
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The lattice perspe

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short elements of
$\mathbf{Z}A+\mathbf{Z}B+\mathbf{Z}C+$
$A=(201,-233,-$
$B=(935,-1063,$
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$D=(718,-829,-$

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The lattice perspective

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short elements of lattice
$\mathbf{Z}A + \mathbf{Z}B + \mathbf{Z}C + \mathbf{Z}D$ where
$A = (201, -233, -430, -71$
$B = (935, -1063, -1986, -$
$C = (979, -1119, -2092, -$
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$D = (718, -829, -1537, -2546)$.

Find $(3, 1, 4, 1)$ as $-37A + 3B - 7C + 16D$.
This was my original g.

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Also find, e.g., $(-4, -1, 3, 1)$.
Multiplying by root of unity (here ζ^2) preserves shortness.

rt-generator problem:
hort" nonzero $g \in \mathcal{O}$
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ubmodule of $\mathcal{O}$ gen by
$33\zeta - 430\zeta^2 - 712\zeta^3$,
$063\zeta - 1986\zeta^2 - 3299\zeta^3$,
$119\zeta - 2092\zeta^2 - 3470\zeta^3$,
$29\zeta - 1537\zeta^2 - 2546\zeta^3$
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$86\zeta^2 - 3299\zeta^3$,
$92\zeta^2 - 3470\zeta^3$,
$7\zeta^2 - 2546\zeta^3$

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Find $(3, 1, 4, 1)$ as $-37A + 3B - 7C + 16D$.
This was my original g.

Also find, e.g., $(-4, -1, 3, 1)$.
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y-smoothness chance $\approx 1/y$ if $\log y \approx \sqrt{(1/2)\log x \log\log x}$.

Variation: Ignore $g\mathcal{O}$.
Generate rather short $\alpha \in \mathcal{O}$,
factor $\alpha\mathcal{O}$ into small primes.
After enough α's,
solve system of equations;
obtain generator for each prime.

— Restrict to a "factor base": e.g., all primes of norm $\le y$.

"But what if $\alpha\mathcal{O}$ doesn't factor into those primes?"

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$2r_2 = \#\{i : \varphi_i(K) \not\subseteq \mathbf{R}\}$.

e.g. $\zeta = \exp(\pi i/256)$, $K = \mathbf{Q}(\zeta)$: images of ζ under ring maps are $\zeta, \zeta^3, \zeta^5, \ldots, \zeta^{511}$.
$r_1 = 0$; $r_2 = 128$; rank 127.

Compute $\operatorname{Log} g u$ as sum of multiples of $\operatorname{Log} \alpha$ for the original α's.

Find elements of $\operatorname{Log} \mathcal{O}^*$ close to $\operatorname{Log} g u$.

There are exactly n distinct ring maps $\varphi_1, \ldots, \varphi_n : K \to \mathbf{C}$.

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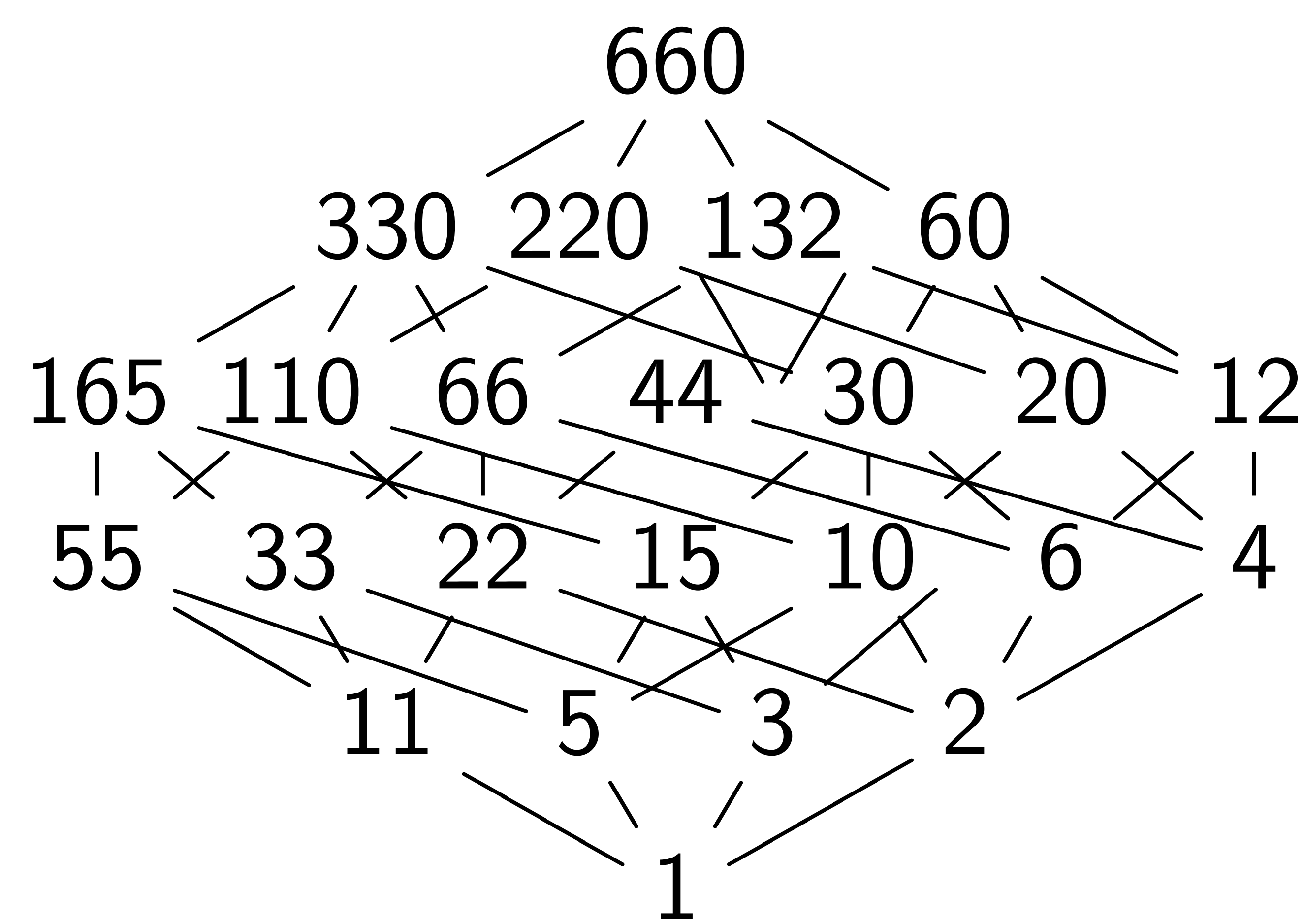

ld-logarithm attack

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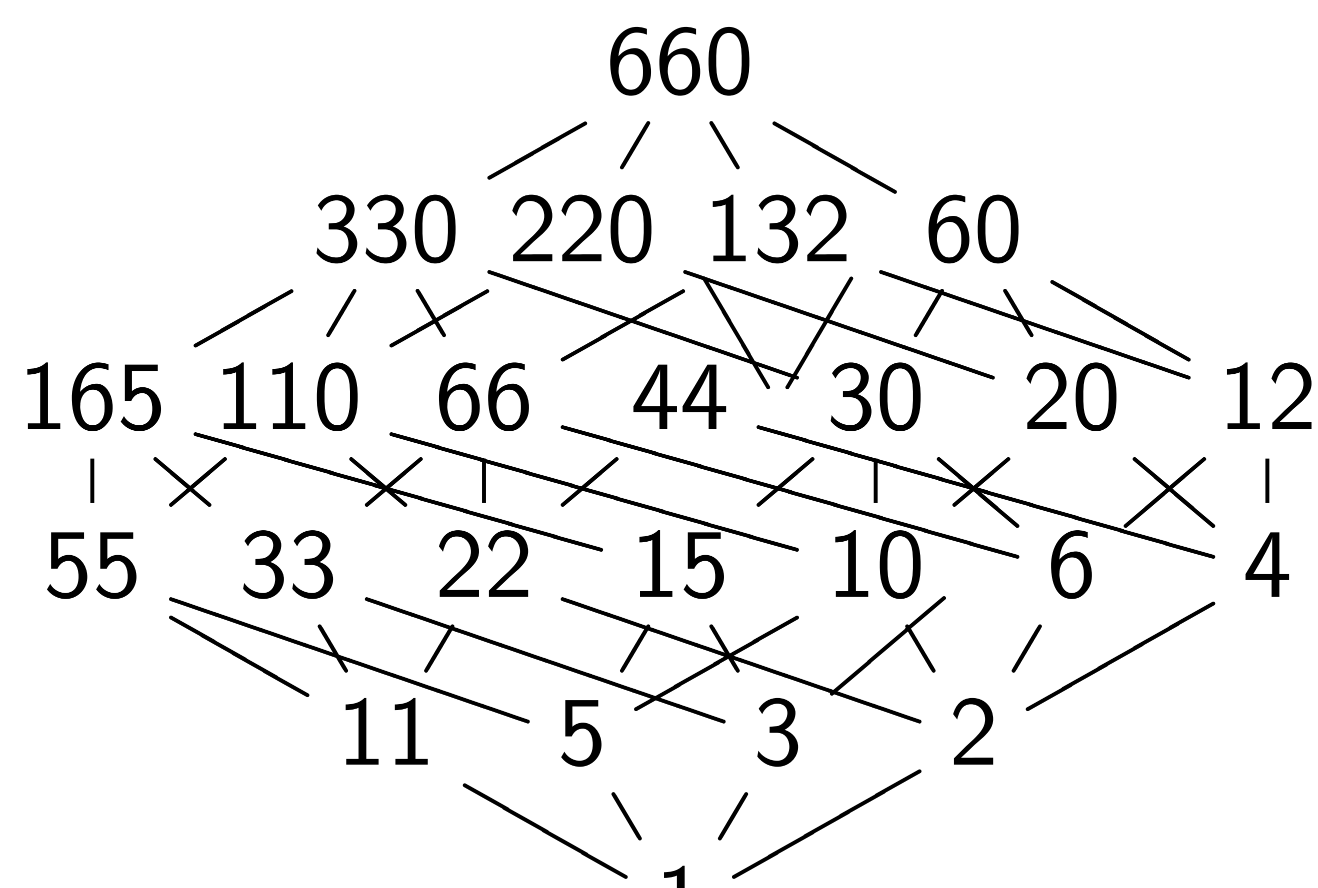

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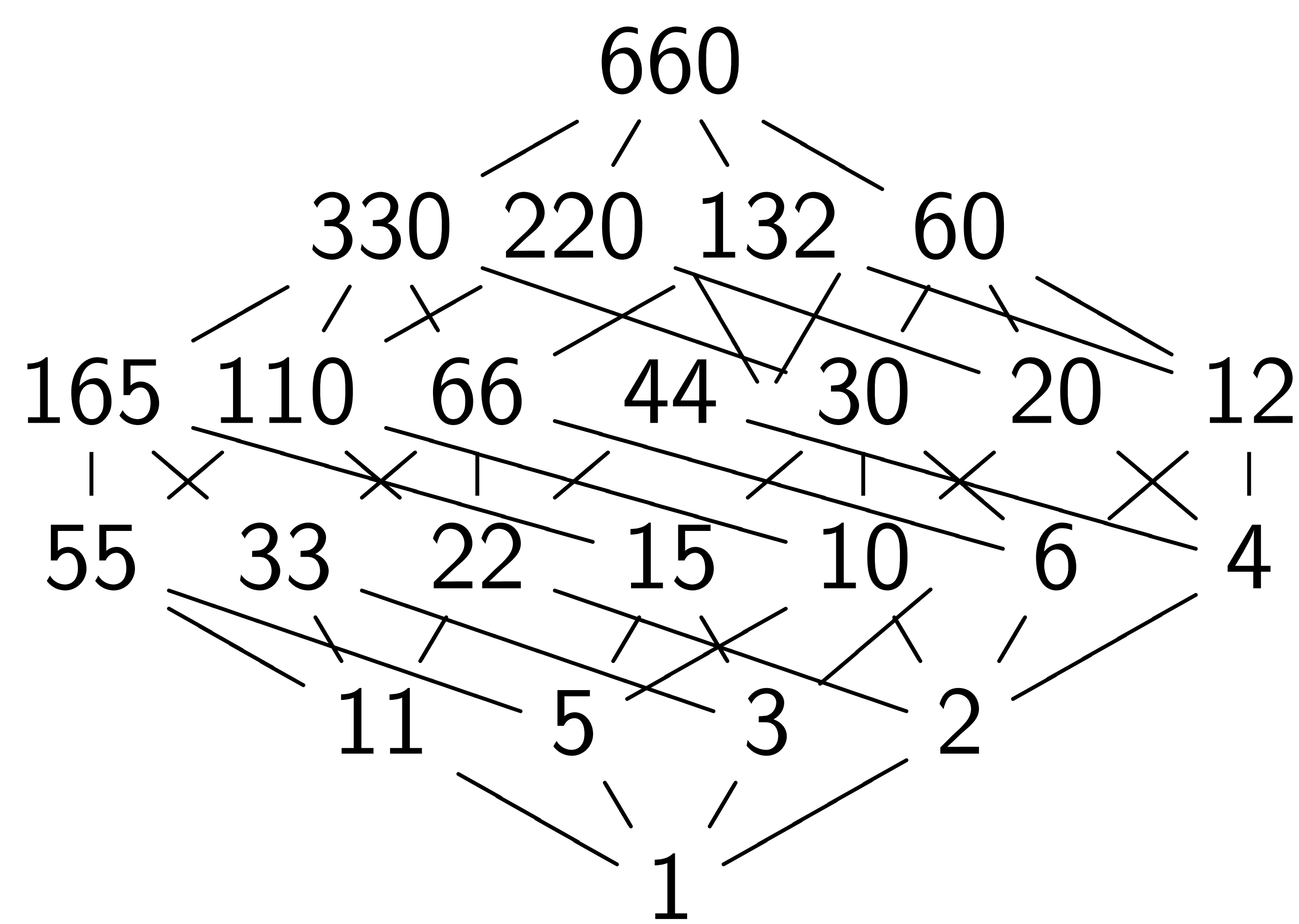

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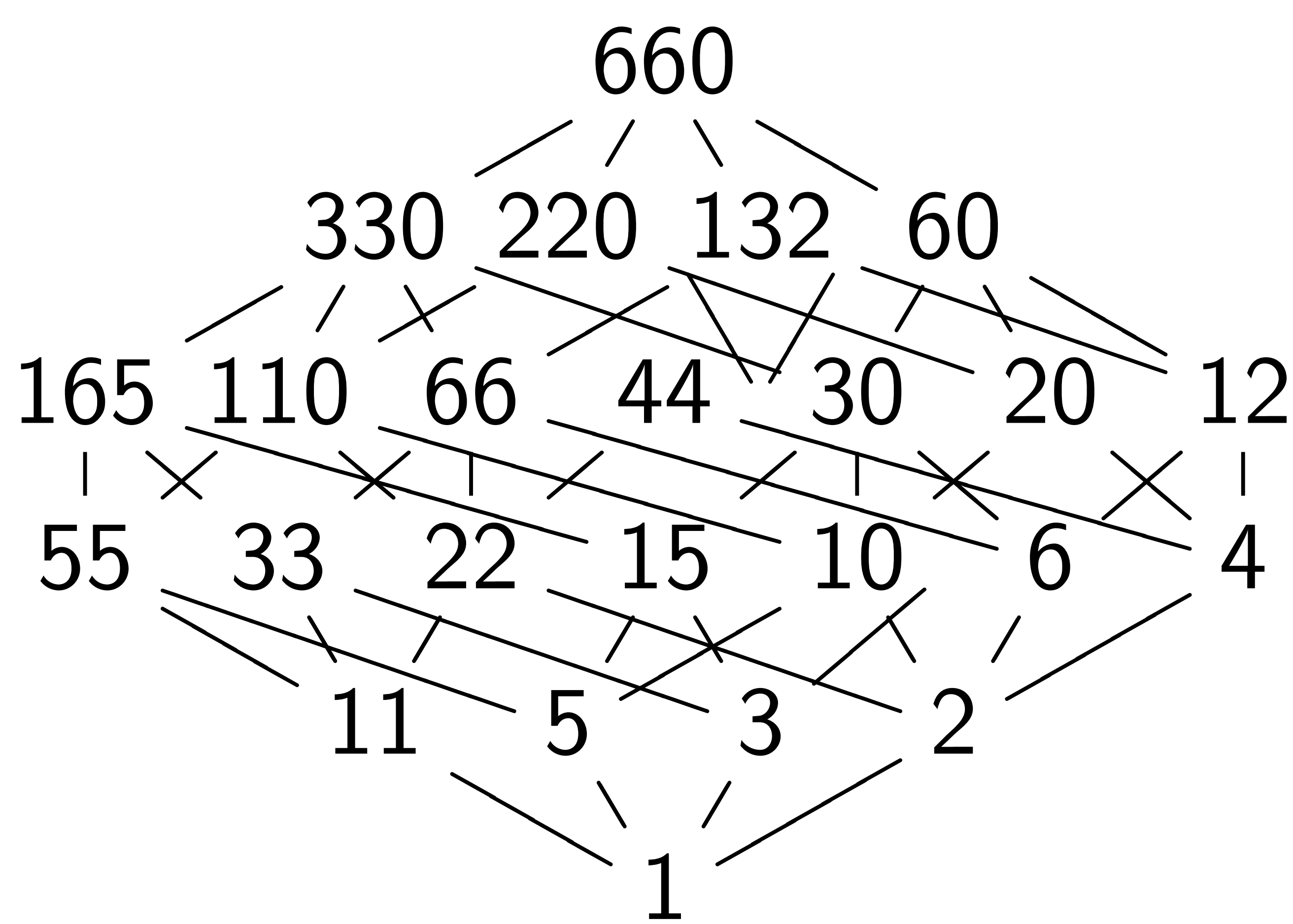

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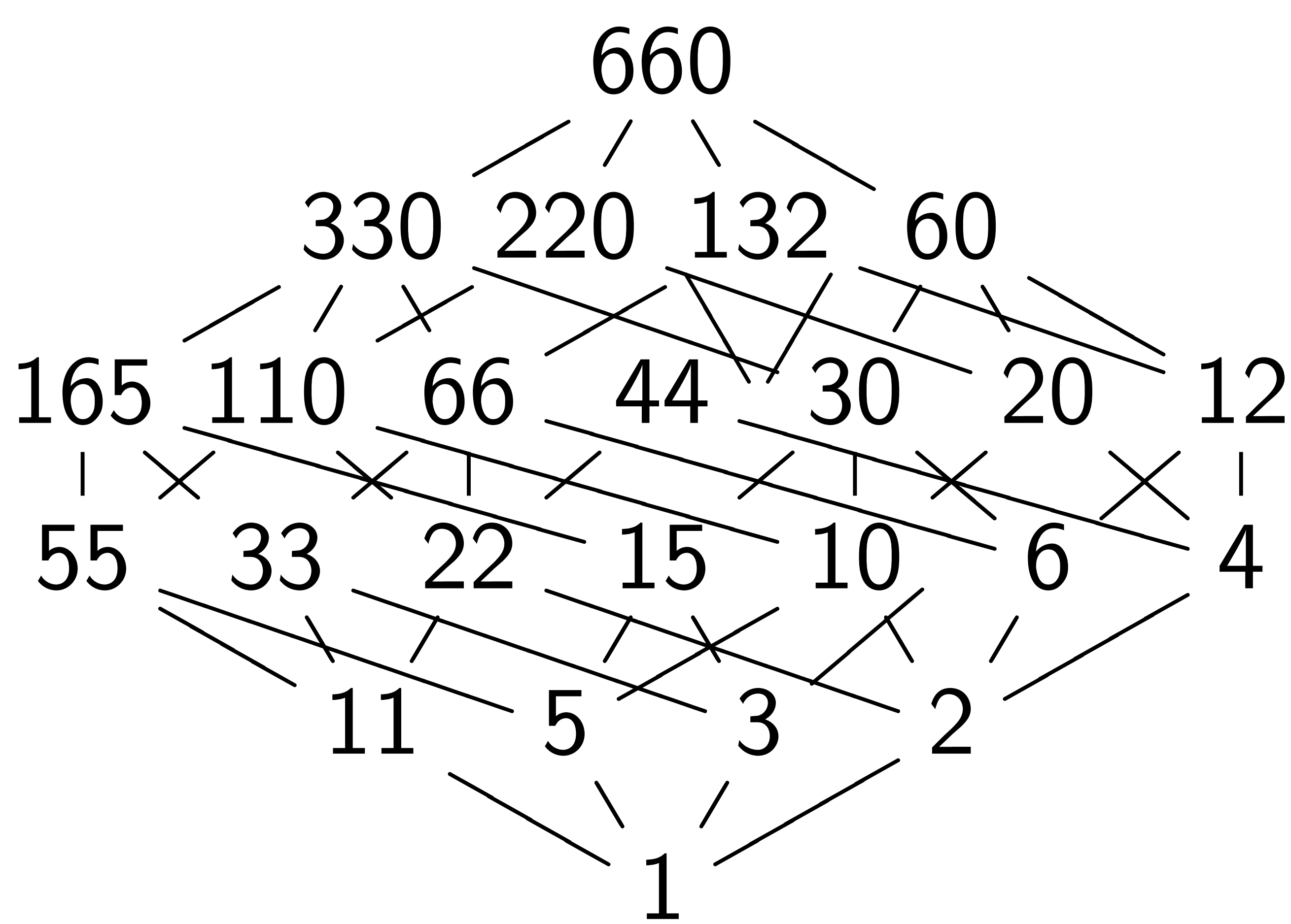

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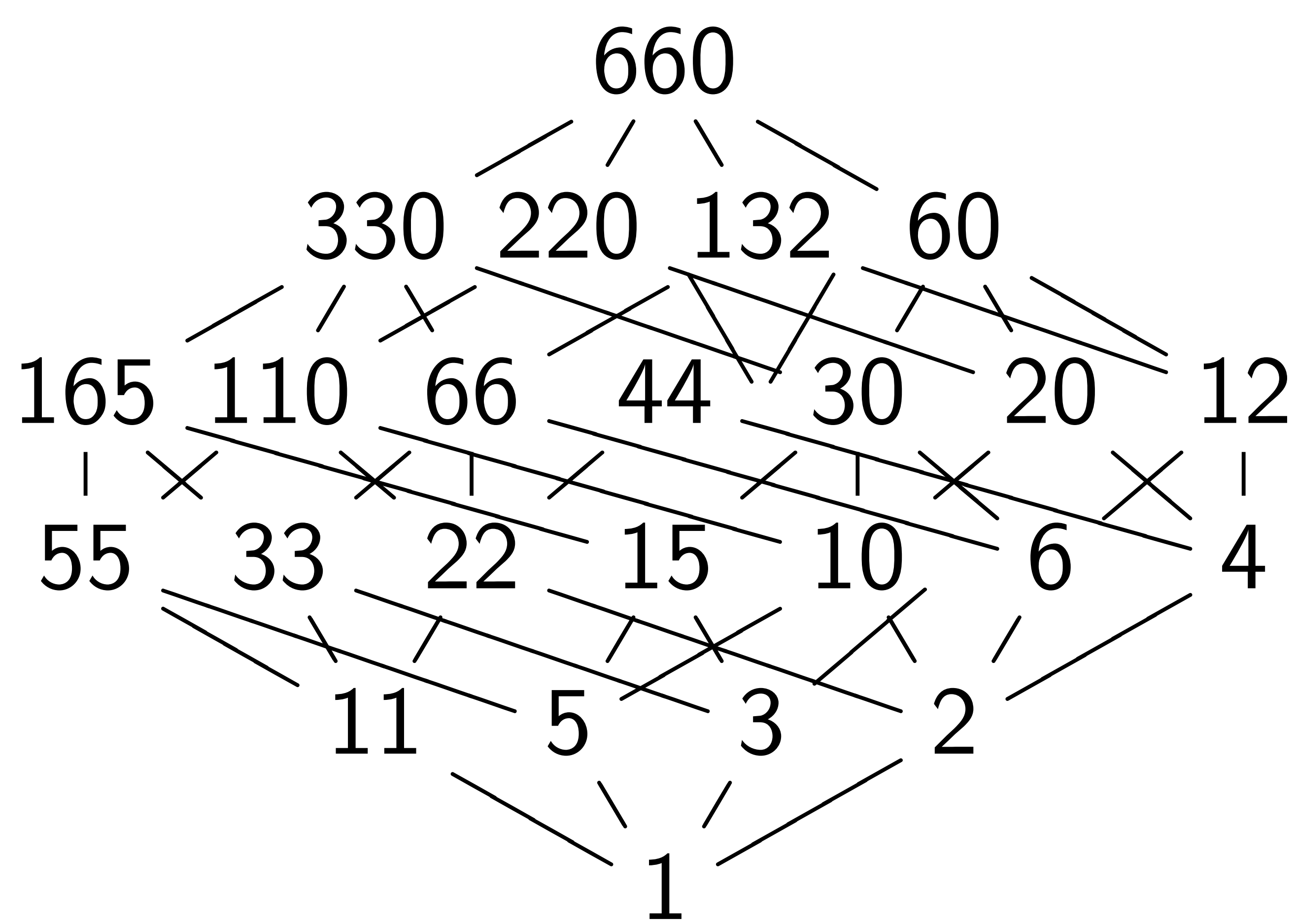

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Many intermediate cases. "Subexponential in *cyclotomic* rings of *highly smooth* index": It's much more general than that.

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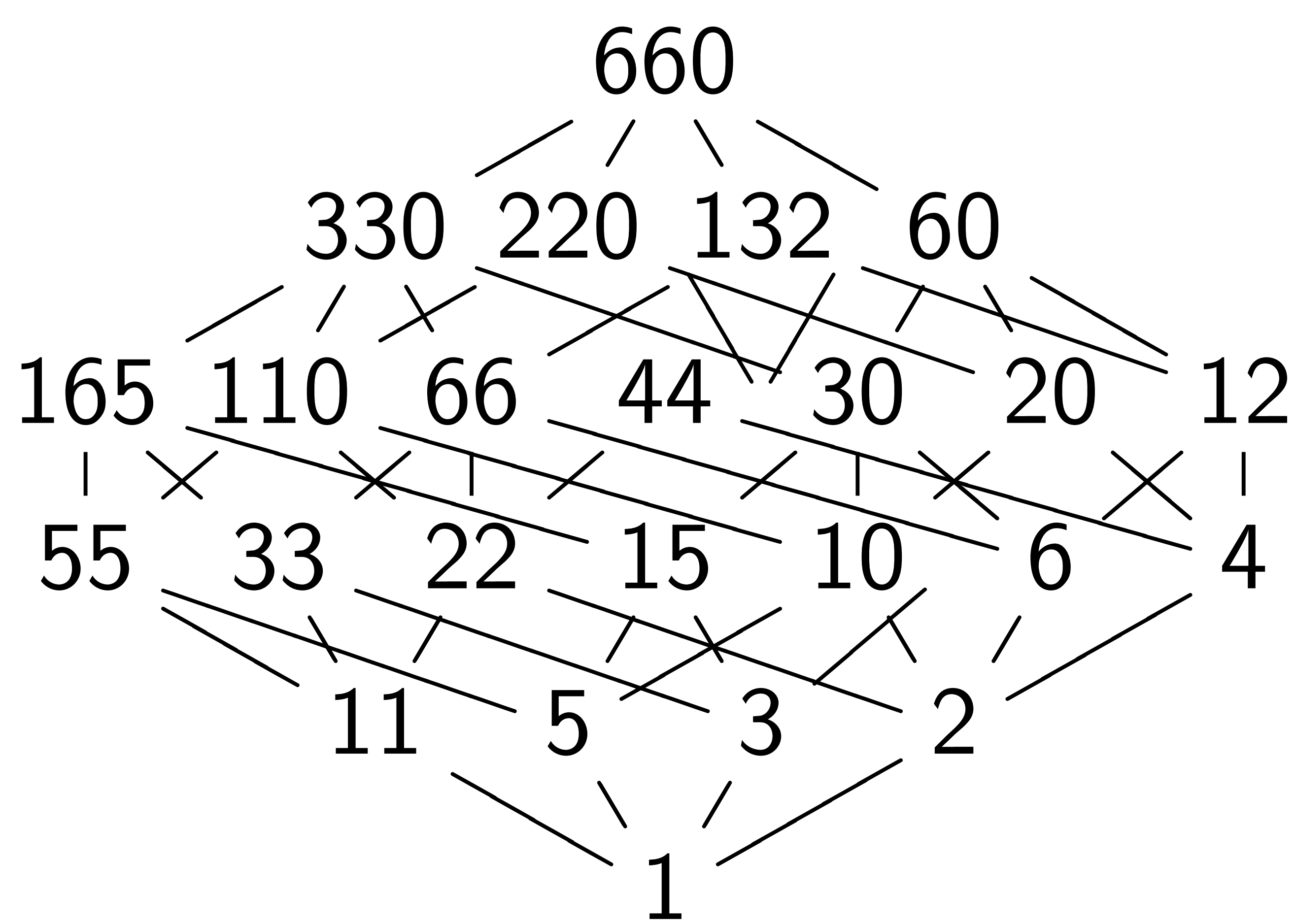

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For cyclotomics this approach is superseded by subsequent Campbell–Groves–Shepherd algorithm, using known (good) basis for cyclotomic units.