A subfield-logarithm attack
against ideal lattices,
part 1: the number-field sieve

D. J. Bernstein
University of Illinois at Chicago &
Technische Universiteit Eindhoven

Sieving small integers $i > 0$
using primes $2, 3, 5, 7$:

1	
2	2
3	3
4	2 2
5	5
6	2 3
7	7
8	2 2 2
9	3 3
10	2 5
11	
12	2 2 3
13	
14	2 7
15	3 5
16	2 2 2 2
17	
18	2 3 3
19	
20	2 2 5

etc.

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13	
14	2 7
15	3 5
16	2 2 2 2
17	
18	2 3 3
19	
20	2 2 5

612	2 2 3 3
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614	2
615	3 5
616	2 2 2 7
617	
618	2 3
619	
620	2 2 5
621	3 3 3
622	2
623	7
624	2 2 2 2 3
625	5 5 5 5
626	2
627	3
628	2 2
629	
630	2 3 3 5 7
631	

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$14 \cdot 625$
$64 \cdot 675$
$75 \cdot 686$

$14 \cdot 64 \cdot$
$= 2^8 3^4 5$

$\gcd\{611$
$= 47.$

$611 = 4$

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Have complete fac
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$64 \cdot 675 = 2^6 3^3 5^2$
$75 \cdot 686 = 2^1 3^1 5^2$

$14 \cdot 64 \cdot 75 \cdot 625 \cdot$
$= 2^8 3^4 5^8 7^4 = (2^4$

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$$= 2^8 3^4 5^8 7^4 = (2^4 3^2 5^4 7^2)^2.$$

$$\gcd\{611, 14 \cdot 64 \cdot 75 - 2^4 3^2 5$$
$$= 47.$$

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No.

By construction 611 divides $s^2 - t^2$
where $s = 14 \cdot 64 \cdot 75$
and $t = 2^4 3^2 5^4 7^2$.
So each prime >7 dividing 611
divides either $s - t$ or $s + t$.

Not terribly surprising
(but not guaranteed in advance!)
that one prime divided $s - t$
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Given long sequence of vectors,
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$1(n+1) = 2^5 3$

$4(n+4) = 2^2 3$

$15(n+15) = 2^1 3$

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$\mathbf{F}_2$-kernel of exponent matrix is
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Plausible conjectu
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Plausible conjecture: **Q** sieve
separate the odd prime divis
of any n, not just 611.

Given n and parameter y:

1. Try to fully factor $i(n +$
into products of primes $\leq y$
for $i \in \{1, 2, 3, \ldots, y^2\}$.

2. Look for nonempty set of
with $i(n + i)$ completely fac
and with $\prod_i i(n + i)$ square.

3. Compute $\gcd\{n, s - t\}$ w
$s = \prod_i i$ and $t = \sqrt{\prod_i i(n +}$

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$15(n + 15) = 2^1 3^1 5^1 7^3$;

$49(n + 49) = 2^4 3^2 5^1 7^2$;

$64(n + 64) = 2^6 3^1 5^1 7^2$.

$\mathbf{F}_2$-kernel of exponent matrix is
gen by (0 1 0 1 1) and (1 0 1 1 0);
e.g., $1(n+1)15(n+15)49(n+49)$
is a square.

Plausible conjecture: **Q** sieve can
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Given n and parameter y:

1. Try to fully factor $i(n + i)$
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2. Look for nonempty set of i's
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3. Compute $\gcd\{n, s - t\}$ where
$s = \prod_i i$ and $t = \sqrt{\prod_i i(n + i)}$.

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$4) = 2^2 3^3 5^2 7^0$;
$15) = 2^1 3^1 5^1 7^3$;
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(This is somewhat pessimistic; smaller numbers usually work.)

Vector length $\approx y/\log y$.
Will there be $> y/\log y$ completely factored congruences out of y^2 congruences?

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If gcd turns out to be 1,
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Conjecturally always works:
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$\exp \sqrt{\left(\frac{1}{2} + o(1)\right) \log n \log \log n}$
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How about
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Given n, try sequence of y's
in geometric progression
until **Q** sieve works;
e.g., increasing powers of 2.

Plausible conjecture: final y
$\exp \sqrt{\left(\frac{1}{2} + o(1)\right) \log n \log \log$
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Increasing c past 1
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reduces linear-algebra cost.
So linear algebra never domi
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How about
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Improving smooth

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replace y^2 with $y^{c+1+o(1)} =$

$\exp \sqrt{\left(\frac{(c+1)^2+o(1)}{2c}\right) \log n \log \log n}$.

Increasing c past 1

increases number of i's but

reduces linear-algebra cost.

So linear algebra never dominates

when y is chosen properly.

Improving smoothness chanc

Smoothness chance of $i(n +$

degrades as i grows.

Smaller for $i \approx y^2$ than for

Crude analysis: $i(n+i)$ gro

$\approx yn$ if $i \approx y$;

$\approx y^2 n$ if $i \approx y^2$.

More careful analysis:

$n + i$ doesn't degrade, but

i is always smooth for $i \leq y$

only 30% chance for $i \approx y^2$.

Can we select congruences

to avoid this degradation?

More generally, if $y \in$
$\exp \sqrt{\left(\frac{1}{2c} + o(1)\right) \log n \log \log n}$,
conjectured y-smoothness chance
is $1/y^{c+o(1)}$.

Find enough smooth congruences
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replace y^2 with $y^{c+1+o(1)} =$
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Improving smoothness chances

Smoothness chance of $i(n+i)$
degrades as i grows.
Smaller for $i \approx y^2$ than for $i \approx y$.

Crude analysis: $i(n+i)$ grows.
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$\frac{1}{2c} + o(1)\big)\log n \log\log n$,

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to avoid this degradation?

Choose q, square
Choose a "q-subla
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$2q - (n \bmod q)$, 3
etc.

Check smoothness
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Choose q, square of large pr
Choose a "q-sublattice" of i
arithmetic progression of i's
where q divides each $i(n+$
e.g. progression $q - (n \bmod$
$2q - (n \bmod q)$, $3q - (n$ mo
etc.

Check smoothness of
generalized congruence $i(n$
for i's in this sublattice.
e.g. check whether $i, (n+i)$
smooth for $i = q - (n \bmod$

Try many large q's.
Rare for i's to overlap.

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where q divides each $i(n+i)$.
e.g. progression $q - (n \bmod q)$,
$2q - (n \bmod q)$, $3q - (n \bmod q)$,
etc.

Check smoothness of
generalized congruence $i(n+i)/q$
for i's in this sublattice.
e.g. check whether $i, (n+i)/q$ are
smooth for $i = q - (n \bmod q)$ etc.

Try many large q's.
Rare for i's to overlap.

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e.g. check whether $i, (n+i)/q$ are
smooth for $i = q - (n \bmod q)$ etc.

Try many large q's.
Rare for i's to overlap.

e.g. $n =$

Original

i	n
1	3
2	3
3	3

Use 997
$i \in 8024$

8024
17964
27904

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for $i \approx y^2$.

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Choose q, square of large prime.
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e.g. check whether $i, (n+i)/q$ are
smooth for $i = q - (n \bmod q)$ etc.

Try many large q's.
Rare for i's to overlap.

e.g. $n = 3141592$

Original **Q** sieve:

i	$n+i$
1	3141592653
2	3141592653
3	3141592653

Use 997^2-sublattic

$i \in 802458 + 994$

i	$(n+$
802458	316
1796467	316
2790476	316

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$i \approx y.$

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Choose q, square of large prime.
Choose a "q-sublattice" of i's:
arithmetic progression of i's
where q divides each $i(n+i)$.
e.g. progression $q-(n \bmod q)$,
$2q-(n \bmod q)$, $3q-(n \bmod q)$,
etc.

Check smoothness of
generalized congruence $i(n+i)/q$
for i's in this sublattice.
e.g. check whether $i, (n+i)/q$ are
smooth for $i = q-(n \bmod q)$ etc.

Try many large q's.
Rare for i's to overlap.

e.g. $n = 31415926535897$

Original **Q** sieve:

i	$n+i$
1	3141592653589732
2	3141592653589732
3	3141592653589732

Use 997^2-sublattice,
$i \in 802458 + 994009\mathbf{Z}$:

i	$(n+i)/997^2$
802458	3160527373
1796467	3160527373
2790476	3160527373

Choose q, square of large prime.
Choose a "q-sublattice" of i's:
arithmetic progression of i's
where q divides each $i(n+i)$.
e.g. progression $q-(n \bmod q)$,
$2q-(n \bmod q)$, $3q-(n \bmod q)$,
etc.

Check smoothness of
generalized congruence $i(n+i)/q$
for i's in this sublattice.
e.g. check whether $i, (n+i)/q$ are
smooth for $i = q-(n \bmod q)$ etc.

Try many large q's.
Rare for i's to overlap.

e.g. $n = 31415926535897932 3$:

Original $\mathbf{Q}$ sieve:

i	$n+i$
1	31415926535897932 4
2	31415926535897932 5
3	31415926535897932 6

Use 997^2-sublattice,
$i \in 802458 + 994009\mathbf{Z}$:

i	$(n+i)/997^2$
802458	316052737309
1796467	316052737310
2790476	316052737311

q, square of large prime.
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divides each $i(n+i)$.
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for $i = q-(n \bmod q)$ etc.

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802458	316052737309
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i	$(n+i)/997^2$
802458	316052737309
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Crude analysis: Su
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For $q \approx n^{1/2}$ have
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e.g. $n = 31415926535897932^3$:

Original $\mathbf{Q}$ sieve:

i	$n+i$
1	31415926535897932**4**
2	31415926535897932**5**
3	31415926535897932**6**

Use 997^2-sublattice,
$i \in 802458 + 994009\mathbf{Z}$:

i	$(n+i)/997^2$
802458	316052737309
1796467	316052737310
2790476	316052737311

Crude analysis: Sublattices
eliminate the growth problem
Have practically unlimited su
of generalized congruences
$(q-(n \bmod q))\dfrac{n+q-(n \bmod$
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More careful analysis: Subla
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For $q \approx n^{1/2}$ have
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$(u/2)^{-u/2}(u/2)^{-u/2} = 2^u/$
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1	31415926535897932 4
2	31415926535897932 5
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Use 997^2-sublattice,
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i	$(n+i)/997^2$
802458	316052737309
1796467	316052737310
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Crude analysis: Sublattices eliminate the growth problem. Have practically unlimited supply of generalized congruences $(q-(n \bmod q))\dfrac{n+q-(n \bmod q)}{q}$ between 0 and n.

More careful analysis: Sublattices are even better than that!
For $q \approx n^{1/2}$ have
$i \approx (n+i)/q \approx n^{1/2} \approx y^{u/2}$
so smoothness chance is roughly
$(u/2)^{-u/2}(u/2)^{-u/2} = 2^u/u^u$,
2^u times larger than before.

= 31415926535897932**3**:

Q sieve:

$n+i$

3141592653589793**24**

3141592653589793**25**

3141592653589793**26**

^{2}-sublattice,

$458 + 994009\mathbf{Z}$:

i	$(n+i)/997^2$
458	316052737309
467	316052737310
476	316052737311

Crude analysis: Sublattices eliminate the growth problem. Have practically unlimited supply of generalized congruences $(q-(n \bmod q))\frac{n+q-(n \bmod q)}{q}$ between 0 and n.

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Even larger improvements from changing polynomial $i($

"Quadratic sieve" (QS) uses $i^2 - n$ with $i \approx \sqrt{n}$; have $i^2 - n \approx n^{1/2+o(1)}$, much smaller than n.

"MPQS" improves $o(1)$ using sublattices: $(i^2 - n)/$ But still $\approx n^{1/2}$.

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Generalizing beyo

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Recall how the **Q**
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Form a square
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$14(625) \cdot 64(675)$
$= 4410000^2$.

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Generalizing beyond **Q**

The **Q** sieve is a special case
the number-field sieve.

Recall how the **Q** sieve
factors 611:

Form a square
as product of $i(i + 611j)$
for several pairs (i, j):
$14(625) \cdot 64(675) \cdot 75(686)$
$= 4410000^2$.

$\gcd\{611, 14 \cdot 64 \cdot 75 - 4410$
$= 47$.

Even larger improvements
from changing polynomial $i(n+i)$.

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$\gcd\{611, 14 \cdot 64 \cdot 75 - 4410000\}$
$= 47$.

The $\mathbf{Q}(\sqrt$
factors 6

Form a s
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for sever
$(-11 +$
$\cdot (3$
$= (112 -$

Compute
$s = (-1$
$t = 112$
$\gcd\{611$

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$(i^2-n)/q$.

ve" (NFS)

Generalizing beyond $\mathbf{Q}$

The $\mathbf{Q}$ sieve is a special case of the number-field sieve.

Recall how the $\mathbf{Q}$ sieve factors 611:

Form a square as product of $i(i+611j)$ for several pairs (i,j):

$14(625)\cdot 64(675)\cdot 75(686)$

$= 4410000^2$.

$\gcd\{611, 14\cdot 64\cdot 75 - 4410000\}$

$= 47$.

The $\mathbf{Q}(\sqrt{14})$ sieve

factors 611 as follo

Form a square

as product of $(i+$

for several pairs $(i$

$(-11+3\cdot 25)(-1$

$\cdot(3+25)(3+$

$= (112-16\sqrt{14})^2$

Compute

$s = (-11+3\cdot 25)$

$t = 112-16\cdot 25,$

$\gcd\{611, s-t\} =$

$(n+i)$.

s

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Generalizing beyond $\mathbf{Q}$

The $\mathbf{Q}$ sieve is a special case of the number-field sieve.

Recall how the $\mathbf{Q}$ sieve factors 611:

Form a square as product of $i(i+611j)$ for several pairs (i,j):

$14(625) \cdot 64(675) \cdot 75(686)$
$= 4410000^2$.

$\gcd\{611, 14 \cdot 64 \cdot 75 - 4410000\}$
$= 47$.

The $\mathbf{Q}(\sqrt{14})$ sieve factors 611 as follows:

Form a square as product of $(i+25j)(i+$ for several pairs (i,j):

$(-11+3\cdot 25)(-11+3\sqrt{14}$
$\cdot (3+25)(3+\sqrt{14})$
$= (112-16\sqrt{14})^2$.

Compute

$s = (-11+3\cdot 25)\cdot(3+25$

$t = 112 - 16\cdot 25$,

$\gcd\{611, s-t\} = 13$.

Generalizing beyond $\mathbf{Q}$

The $\mathbf{Q}$ sieve is a special case of the number-field sieve.

Recall how the $\mathbf{Q}$ sieve factors 611:

Form a square as product of $i(i + 611j)$ for several pairs (i, j):

$14(625) \cdot 64(675) \cdot 75(686)$
$= 4410000^2$.

$\gcd\{611, 14 \cdot 64 \cdot 75 - 4410000\}$
$= 47$.

The $\mathbf{Q}(\sqrt{14})$ sieve factors 611 as follows:

Form a square as product of $(i + 25j)(i + \sqrt{14}j)$ for several pairs (i, j):

$(-11 + 3 \cdot 25)(-11 + 3\sqrt{14})$
$\cdot (3 + 25)(3 + \sqrt{14})$
$= (112 - 16\sqrt{14})^2$.

Compute

$s = (-11 + 3 \cdot 25) \cdot (3 + 25)$,

$t = 112 - 16 \cdot 25$,

$\gcd\{611, s - t\} = 13$.

zing beyond **Q**

sieve is a special case of
ber-field sieve.

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611:

square
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$\cdot 64(675)\cdot 75(686)$
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$, 14\cdot 64\cdot 75 - 4410000\}$

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$$(-11+3\cdot 25)(-11+3\sqrt{14})$$
$$\cdot(3+25)(3+\sqrt{14})$$
$$=(112-16\sqrt{14})^2.$$

Compute
$s=(-11+3\cdot 25)\cdot(3+25)$,
$t=112-16\cdot 25$,
$\gcd\{611, s-t\}=13$.

Why do

Answer:
$\mathbf{Z}[\sqrt{14}]$
since 25

Apply ri
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$\cdot(3$
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i.e. $s^2=$

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$+ 611j)$
$, j)$:
$\cdot 75(686)$

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Compute
$s = (-11 + 3 \cdot 25) \cdot (3 + 25),$
$t = 112 - 16 \cdot 25,$
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Why does this wo

Answer: Have ring
$\mathbf{Z}[\sqrt{14}] \to \mathbf{Z}/611,$
since $25^2 = 14$ in

Apply ring morphi
$(-11 + 3 \cdot 25)(-1$
$\cdot (3 + 25)(3 +$
$= (112 - 16 \cdot 25)^2$

i.e. $s^2 = t^2$ in $\mathbf{Z}/6$

Unsurprising to fin

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The $\mathbf{Q}(\sqrt{14})$ sieve
factors 611 as follows:

Form a square
as product of $(i + 25j)(i + \sqrt{14}j)$
for several pairs (i, j):
$(-11 + 3 \cdot 25)(-11 + 3\sqrt{14})$
$\cdot (3 + 25)(3 + \sqrt{14})$
$= (112 - 16\sqrt{14})^2$.

Compute
$s = (-11 + 3 \cdot 25) \cdot (3 + 25)$,
$t = 112 - 16 \cdot 25$,
$\gcd\{611, s - t\} = 13$.

Why does this work?

Answer: Have ring morphism
$\mathbf{Z}[\sqrt{14}] \to \mathbf{Z}/611$, $\sqrt{14} \mapsto 2$
since $25^2 = 14$ in $\mathbf{Z}/611$.

Apply ring morphism to squ
$(-11 + 3 \cdot 25)(-11 + 3 \cdot 25)$
$\cdot (3 + 25)(3 + 25)$
$= (112 - 16 \cdot 25)^2$ in $\mathbf{Z}/611$

i.e. $s^2 = t^2$ in $\mathbf{Z}/611$.

Unsurprising to find factor.

The $\mathbf{Q}(\sqrt{14})$ sieve
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Form a square
as product of $(i + 25j)(i + \sqrt{14}j)$
for several pairs (i, j):
$(-11 + 3 \cdot 25)(-11 + 3\sqrt{14})$
$\cdot (3 + 25)(3 + \sqrt{14})$
$= (112 - 16\sqrt{14})^2$.

Compute
$s = (-11 + 3 \cdot 25) \cdot (3 + 25)$,
$t = 112 - 16 \cdot 25$,
$\gcd\{611, s - t\} = 13$.

Why does this work?

Answer: Have ring morphism
$\mathbf{Z}[\sqrt{14}] \to \mathbf{Z}/611$, $\sqrt{14} \mapsto 25$,
since $25^2 = 14$ in $\mathbf{Z}/611$.

Apply ring morphism to square:
$(-11 + 3 \cdot 25)(-11 + 3 \cdot 25)$
$\cdot (3 + 25)(3 + 25)$
$= (112 - 16 \cdot 25)^2$ in $\mathbf{Z}/611$.

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$- 16\sqrt{14})^2$.

e
$1 + 3 \cdot 25) \cdot (3 + 25)$,
$- 16 \cdot 25$,
$, s - t\} = 13$.

Why does this work?

Answer: Have ring morphism
$\mathbf{Z}[\sqrt{14}] \to \mathbf{Z}/611$, $\sqrt{14} \mapsto 25$,
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$\cdot (3 + 25)(3 + 25)$
$= (112 - 16 \cdot 25)^2$ in $\mathbf{Z}/611$.

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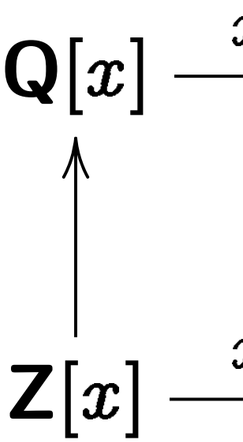
Unsurprising to find factor.

Diagram



$\mathbf{Z}[x]$ use
$\{i_0 x^0 +$
$\mathbf{Z}[\sqrt{14}]$
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$\mathbf{Z}/611$ u
on $\{0, 1,$

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$25j)(i+\sqrt{14}j)$

$,j)$:

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13.

Why does this work?

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i.e. $s^2 = t^2$ in $\mathbf{Z}/611$.

Unsurprising to find factor.

Diagram of ring m

$$\begin{array}{ccc} \mathbf{Q}[x] & \xrightarrow{x\mapsto\sqrt{14}} & \mathbf{Q} \\ \uparrow & & \\ \mathbf{Z}[x] & \xrightarrow{x\mapsto\sqrt{14}} & \mathbf{Z} \end{array}$$

$\mathbf{Z}$

$\mathbf{Z}[x]$ uses poly arit
$\{i_0x^0+i_1x^1+\cdots$
$\mathbf{Z}[\sqrt{14}]$ uses $\mathbf{R}$ ari
$\{i_0+i_1\sqrt{14} : i_0,$
$\mathbf{Z}/611$ uses arithm
on $\{0,1,\ldots,610\}$

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Why does this work?

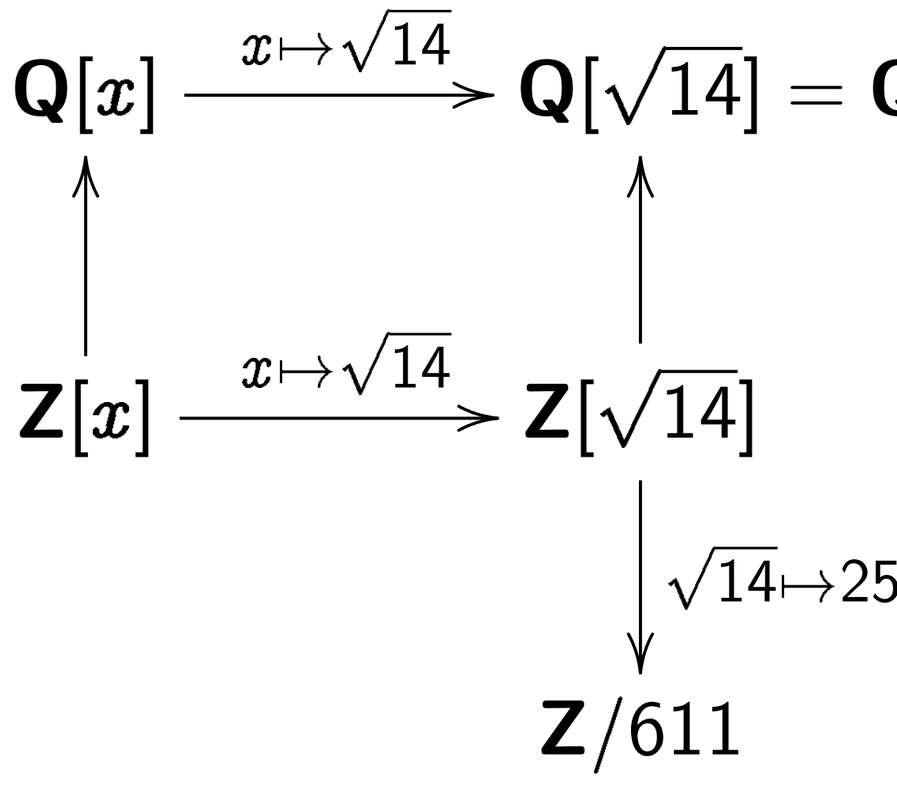
Answer: Have ring morphism
$\mathbf{Z}[\sqrt{14}] \to \mathbf{Z}/611$, $\sqrt{14} \mapsto 25$,
since $25^2 = 14$ in $\mathbf{Z}/611$.

Apply ring morphism to square:
$(-11 + 3 \cdot 25)(-11 + 3 \cdot 25)$
$\cdot (3 + 25)(3 + 25)$
$= (112 - 16 \cdot 25)^2$ in $\mathbf{Z}/611$.

i.e. $s^2 = t^2$ in $\mathbf{Z}/611$.

Unsurprising to find factor.

Diagram of ring morphisms:



$\mathbf{Z}[x]$ uses poly arithmetic on
$\{i_0 x^0 + i_1 x^1 + \cdots : \text{all } i_m \in$
$\mathbf{Z}[\sqrt{14}]$ uses $\mathbf{R}$ arithmetic on
$\{i_0 + i_1\sqrt{14} : i_0, i_1 \in \mathbf{Z}\}$;
$\mathbf{Z}/611$ uses arithmetic mod
on $\{0, 1, \ldots, 610\}$.

Why does this work?

Answer: Have ring morphism
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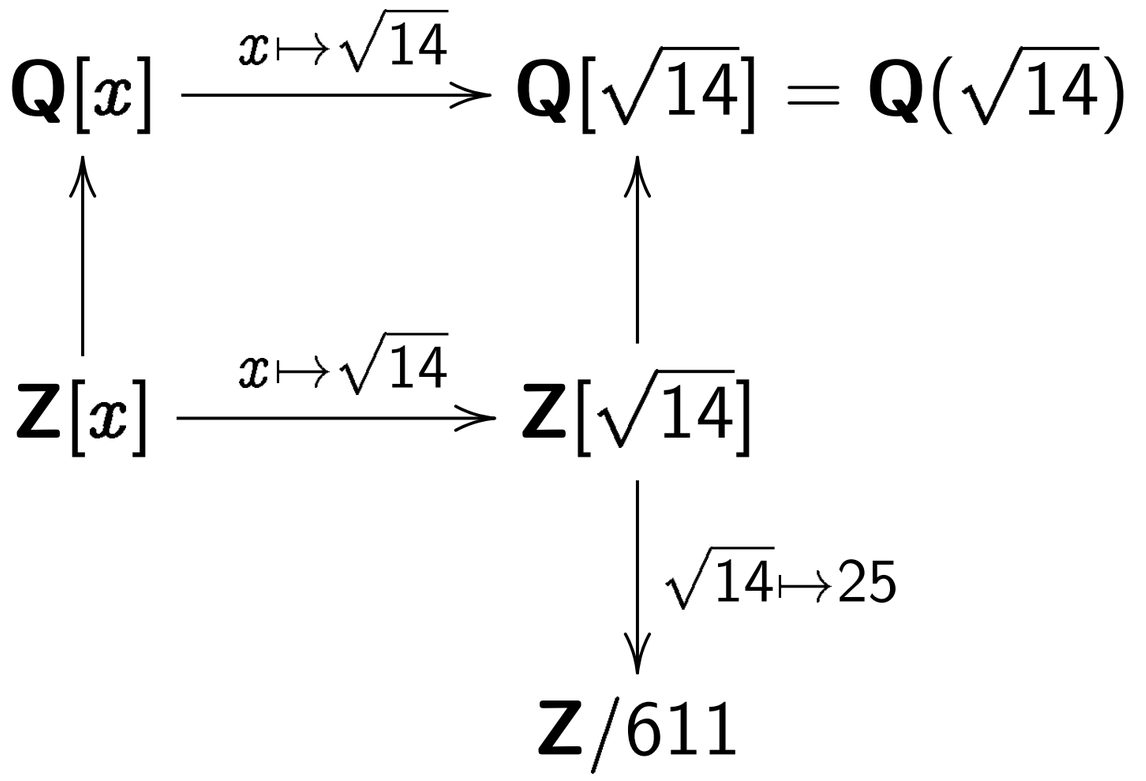
Apply ring morphism to square:

$$(-11 + 3 \cdot 25)(-11 + 3 \cdot 25)$$
$$\cdot (3 + 25)(3 + 25)$$
$$= (112 - 16 \cdot 25)^2 \text{ in } \mathbf{Z}/611.$$

i.e. $s^2 = t^2$ in $\mathbf{Z}/611$.

Unsurprising to find factor.

Diagram of ring morphisms:



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es this work?

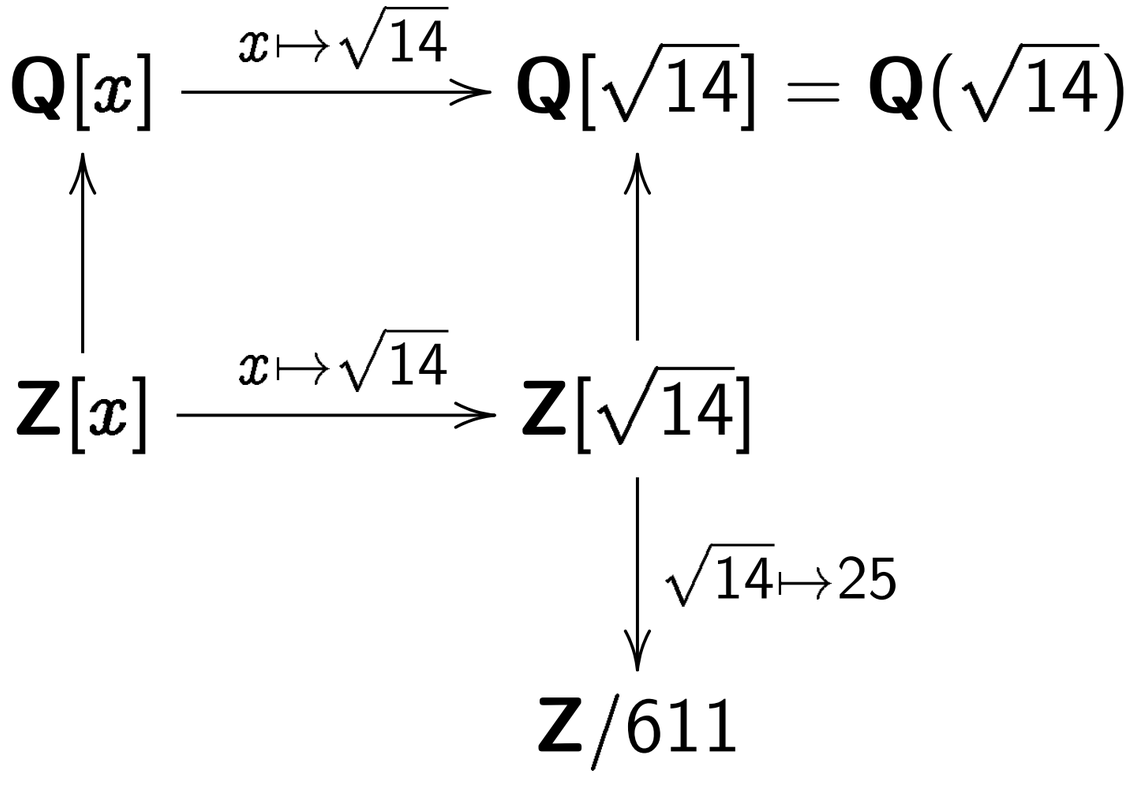
Have ring morphism
$\to \mathbf{Z}/611$, $\sqrt{14} \mapsto 25$,
$^2 = 14$ in $\mathbf{Z}/611$.

ng morphism to square:
$3 \cdot 25)(-11 + 3 \cdot 25)$
$3 + 25)(3 + 25)$
$- 16 \cdot 25)^2$ in $\mathbf{Z}/611$.

$= t^2$ in $\mathbf{Z}/611$.

ising to find factor.

Diagram of ring morphisms:



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Generali
to $(f, m$
$m \in \mathbf{Z}$,

Write d
$f = f_d x$

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Pick $\alpha \in$
Then f_d
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$\sqrt{14} \mapsto 25$,
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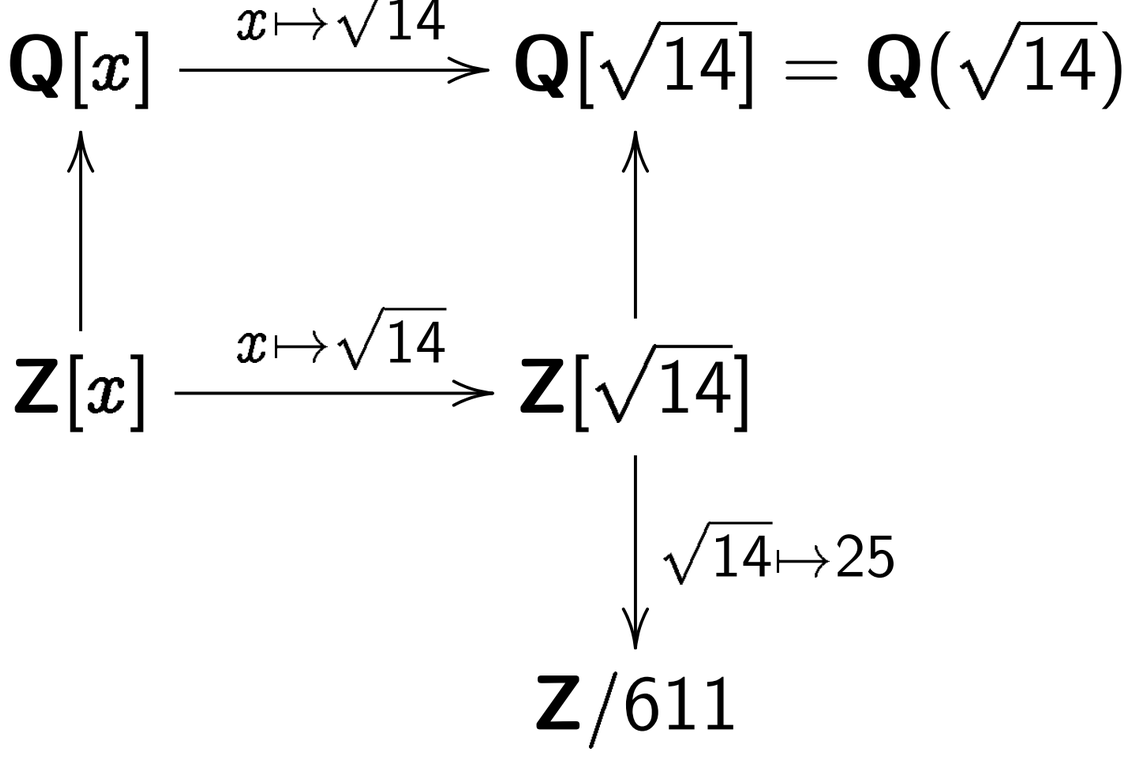
Diagram of ring morphisms:



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Generalize from $(x$
to (f, m) with irre
$m \in \mathbf{Z}$, $f(m) \in n$

Write $d = \deg f$,
$f = f_d x^d + \cdots +$

Can take $f_d = 1$ f
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better parameter s

Pick $\alpha \in \mathbf{C}$, root
Then $f_d \alpha$ is a roo
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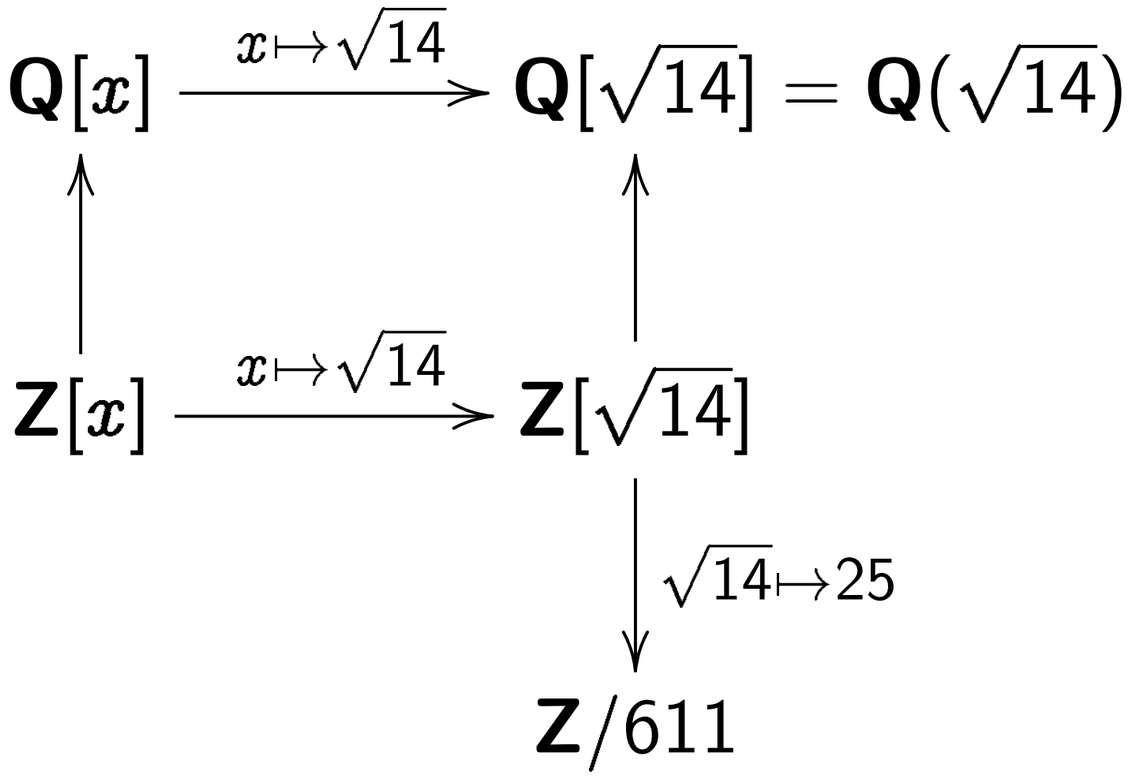
Diagram of ring morphisms:



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Generalize from $(x^2 - 14, 25$
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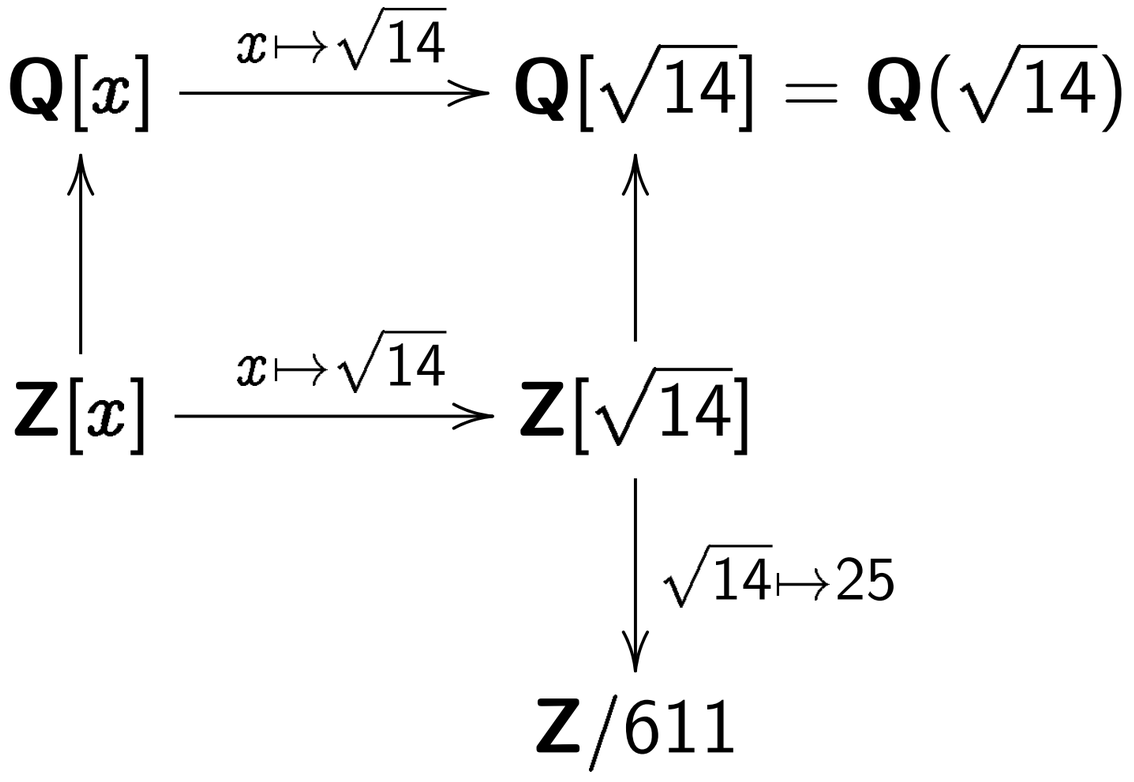

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Can take $f_d = 1$ for simplicity,
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Pick $\alpha \in \mathbf{C}$, root of f.
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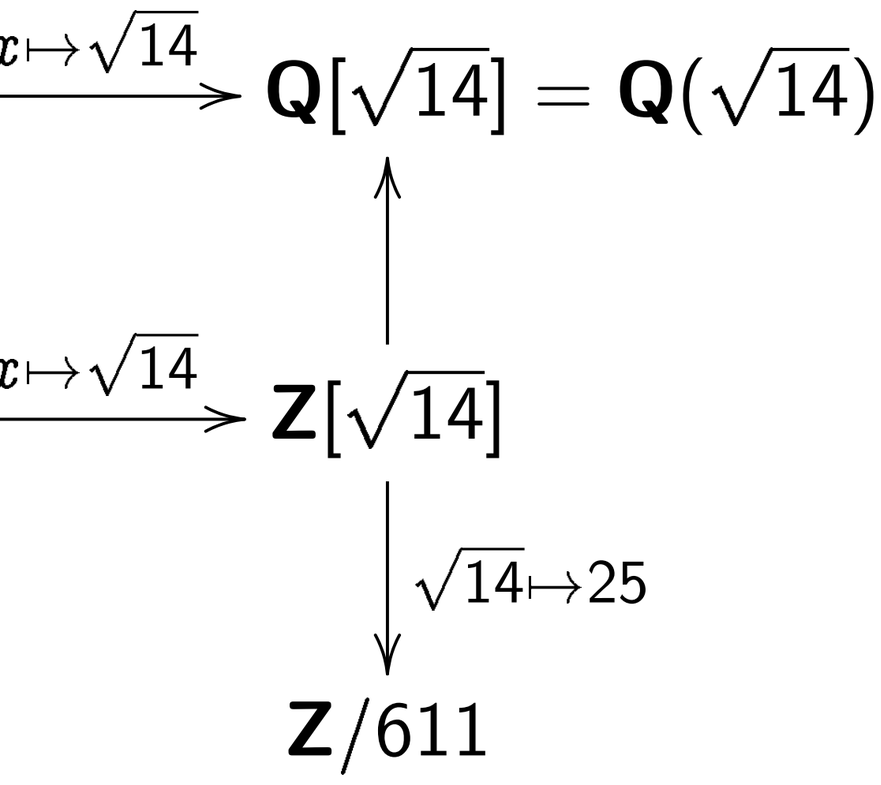
n of ring morphisms:



es poly arithmetic on
$i_1 x^1 + \cdots$: all $i_m \in \mathbf{Z}\}$;
uses $\mathbf{R}$ arithmetic on
$\sqrt{14} : i_0, i_1 \in \mathbf{Z}\}$;
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Generalize from $(x^2 - 14, 25)$
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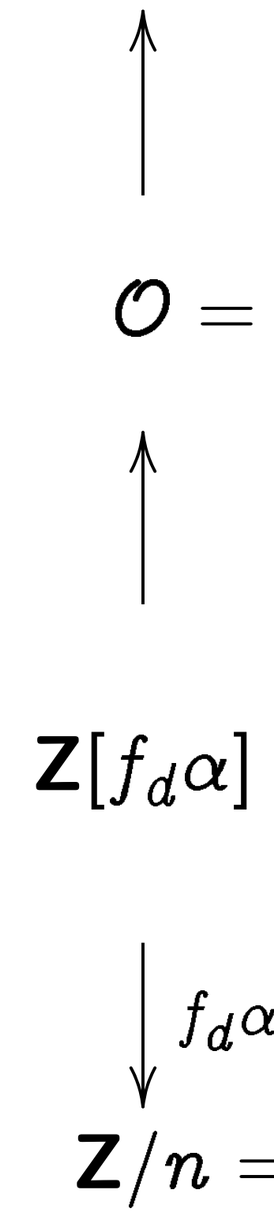

norphisms:

$[\sqrt{14}] = \mathbf{Q}(\sqrt{14})$

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$[\sqrt{14}]$

$\downarrow \sqrt{14} \mapsto 25$

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Generalize from $(x^2 - 14, 25)$
to (f, m) with irred $f \in \mathbf{Z}[x]$,
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$\mathbf{Q}(\alpha) = \{ r_0 + r$
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$r_0, \ldots$

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in $\mathbf{Q}$

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$\mathbf{Z}[f_d\alpha] = \{ i_0 +$
$\cdots +$
$i_0, \ldots$

$\downarrow f_d\alpha \mapsto f_d m$

$\mathbf{Z}/n = \{0, 1, \ldots$

$\mathbf{Q}(\sqrt{14})$

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611

Generalize from $(x^2 - 14, 25)$
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$$\mathbf{Q}(\alpha) = \begin{cases} r_0 + r_1\alpha + r_2\alpha \\ \cdots + r_{d-1}\alpha^{d-1} \\ r_0, \ldots, r_{d-1} \in \mathbf{Q} \end{cases}$$

$\uparrow$

$$\mathcal{O} = \begin{cases} \text{algebraic integers} \\ \text{in } \mathbf{Q}(\alpha) \end{cases}$$

$\uparrow$

$$\mathbf{Z}[f_d\alpha] = \begin{cases} i_0 + i_1 f_d \alpha + \\ \cdots + i_{d-1} f_d^{d-1} \\ i_0, \ldots, i_{d-1} \in \mathbf{Z} \end{cases}$$

$\downarrow f_d\alpha \mapsto f_d m$

$$\mathbf{Z}/n = \{0, 1, \ldots, n-1\}$$

Generalize from $(x^2 - 14, 25)$
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$$\uparrow$$

$$\mathcal{O} = \left\{ \begin{array}{c} \text{algebraic integers} \\ \text{in } \mathbf{Q}(\alpha) \end{array} \right\}$$

$$\uparrow$$

$$\mathbf{Z}[f_d\alpha] = \left\{ \begin{array}{l} i_0 + i_1 f_d\alpha + \\ \cdots + i_{d-1} f_d^{d-1} \alpha^{d-1}: \\ i_0, \ldots, i_{d-1} \in \mathbf{Z} \end{array} \right\}$$

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$$\mathbf{Z}/n = \{0, 1, \ldots, n-1\}$$

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$(x/f_d) \in \mathbf{Z}[x]$.

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Build square in $\mathbf{Q}($

congruences $(i - j$

with $i\mathbf{Z} + j\mathbf{Z} = \mathbf{Z}$

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$$\mathcal{O} = \left\{ \begin{array}{c} \text{algebraic integers} \\ \text{in } \mathbf{Q}(\alpha) \end{array} \right\}$$

$\uparrow$

$$\mathbf{Z}[f_d\alpha] = \left\{ \begin{array}{l} i_0 + i_1 f_d \alpha + \\ \cdots + i_{d-1} f_d^{d-1} \alpha^{d-1} : \\ i_0, \ldots, i_{d-1} \in \mathbf{Z} \end{array} \right\}$$

$\downarrow f_d\alpha \mapsto f_d m$

$$\mathbf{Z}/n = \{0, 1, \ldots, n-1\}$$

Build square in $\mathbf{Q}(\alpha)$ from congruences $(i - jm)(i - j$ with $i\mathbf{Z} + j\mathbf{Z} = \mathbf{Z}$ and $j > 0$

Could replace $i - jx$ by higher-deg irred in $\mathbf{Z}[x]$; quadratics seem fairly small for some number fields. But let's not bother.

Say we have a square $\prod_{(i,j)\in S}(i - jm)(i - j\alpha)$ in $\mathbf{Q}(\alpha)$; now what?

$$\mathbf{Q}(\alpha) = \left\{ \begin{array}{l} r_0 + r_1\alpha + r_2\alpha^2 + \\ \cdots + r_{d-1}\alpha^{d-1}; \\ r_0, \ldots, r_{d-1} \in \mathbf{Q} \end{array} \right\}$$

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In $\mathbf{Z}/n$ have $\varphi(r)^2 =$
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How to find square product
of congruences $(i - jm)(i -$

Start with congruences for,
e.g., y^2 pairs (i, j).

Look for y-smooth congruen
y-smooth $i - jm$ and
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$f_d i^d + \cdots + f_0 j^d = j^d f(i/j$

Find enough smooth congru
Perform linear algebra on
exponent vectors mod 2.

$\prod(i - jm)(i - j\alpha)f_d^2$
is a square in $\mathcal{O}$,
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Multiply by $g'(f_d\alpha)^2$,
putting square root into $\mathbf{Z}[f_d\alpha]$:
compute r with $r^2 = g'(f_d\alpha)^2$.
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Find enough smooth congruences. Perform linear algebra on exponent vectors mod 2.

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Find enough smooth congruences.
Perform linear algebra on exponent vectors mod 2.

Exponent vectors
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Value $\operatorname{ord}_p(i - jm$

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Exponent vectors have
many "rational" component
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One rational component
for each prime $p \leq y$.
Value $\operatorname{ord}_p(i - jm)$.

One rational component for
Value 0 if $i - jm > 0$,
value 1 if $i - jm < 0$.

If $\prod(i - jm)$ is a square
then vectors add to 0
in rational components.

How to find square product
of congruences $(i - jm)(i - j\alpha)$?

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Value 0 if $i - jr \notin$
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Value 0 if $i - jr \notin p\mathbf{Z}$;
otherwise $\text{ord}_p(j^d f(i/j))$.

This is the same as
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at the prime $p\mathcal{O} + (f_d\alpha - f$
Recall that $i\mathbf{Z} + j\mathbf{Z} = \mathbf{Z}$,
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Recall that $i\mathbf{Z} + j\mathbf{Z} = \mathbf{Z}$,
so no higher-degree primes.

One character component
for each pair (p, r) with
p in a short range above y.

Value 0 if $i - jr$ is a
square in $\mathbf{F}_p$, else 1.

If $\prod(i - j\alpha)$ is a square
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One algebraic component
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One character component
for each pair (p, r) with
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Value 0 if $i - jr$ is a
square in $\mathbf{F}_p$, else 1.

If $\prod(i - j\alpha)$ is a square
then vectors add to 0
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"$(f_d \operatorname{disc} f)$-Selmer group."

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Consider a sublattice
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Assume squarish lattice.
$(i - jm)j^d f(i/j)$
expands by factor $q^{(d+1)/2}$
before division by q.

Number of sublattice elemen
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Compared to just using $q =$
conjecturally obtain $y^{4/(d+1)}$
times as many congruences
by using sublattices for
all y-smooth integers $q \leq y^2$

Separately consider
$i - jm$ and $j^d f(i/j)/q$
for more precise analysis.

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Sublattices

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on $(i - jm)j^d f(i/j)$
is proportional to $q^{-(d-1)/(d+1)}$.

Compared to just using $q = 1$,
conjecturally obtain $y^{4/(d+1)+o(1)}$
times as many congruences
by using sublattices for
all y-smooth integers $q \leq y^2$.

Separately consider
$i - jm$ and $j^d f(i/j)/q$
for more precise analysis.

Limit congruences accordingly,
increasing smoothness chances.

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Limit congruences accordingly,
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Multiple number f

Assume that $f +$
is also irred.

Pick $\beta \in \mathbf{C}$, root
Two congruences
$(i - jm)(i - j\alpha)$; (
Expand exponent
handle both $\mathbf{Q}(\alpha)$

Merge smoothness
by testing $i - jm$
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Can use many nu
$f + 2(x - m)$ etc.

Compared to just using $q = 1$,
conjecturally obtain $y^{4/(d+1)+o(1)}$
times as many congruences
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all y-smooth integers $q \leq y^2$.

Separately consider
$i - jm$ and $j^d f(i/j)/q$
for more precise analysis.

Limit congruences accordingly,
increasing smoothness chances.

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Multiple number fields

Assume that $f + x - m \in \mathbf{Z}$
is also irred.

Pick $\beta \in \mathbf{C}$, root of $f + x -$
Two congruences for (i, j):
$(i - jm)(i - j\alpha)$; $(i - jm)(i$
Expand exponent vectors to
handle both $\mathbf{Q}(\alpha)$ and $\mathbf{Q}(\beta$

Merge smoothness tests
by testing $i - jm$ first,
aborting if $i - jm$ not smoo

Can use many number fields
$f + 2(x - m)$ etc.

Compared to just using $q = 1$,
conjecturally obtain $y^{4/(d+1)+o(1)}$
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all y-smooth integers $q \leq y^2$.

Separately consider
$i - jm$ and $j^d f(i/j)/q$
for more precise analysis.

Limit congruences accordingly,
increasing smoothness chances.

Multiple number fields

Assume that $f + x - m \in \mathbf{Z}[x]$
is also irred.

Pick $\beta \in \mathbf{C}$, root of $f + x - m$.
Two congruences for (i, j):
$(i - jm)(i - j\alpha)$; $(i - jm)(i - j\beta)$.
Expand exponent vectors to
handle both $\mathbf{Q}(\alpha)$ and $\mathbf{Q}(\beta)$.

Merge smoothness tests
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