Hyper-and-elliptic-curve cryptography

(which is not the same as: hyperelliptic-curve cryptography and elliptic-curve cryptography)

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But first some context. . .

ECC security vs. ECDL security

Crypto view of ECDL problem:
Fix finite k, E/k, $P \in E(k)$.
Secret key: random $a \in \mathbf{Z}/\#\mathbf{Z}P$.
Public key: aP.
The ECDL problem: compute secret key from public key.

ECDL solution $\Rightarrow$ ECC attack.

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ECC attack $\Rightarrow$ ECDL solution?
Not necessarily!

Let's look at some examples.

Example 1: **Kummer-line ECDH** (1985 Miller). Bob has secret b; receives $X(A)$ from Alice; uses easy formulas to compute $X(bA)$; encrypts using $X(bA)$.

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Typically Bob checks $X(A) \in k$ but doesn't check $A \in E(k)$. Formulas also work for $A \in E'(k)$ for appropriate twist E' of E. Typically $\#E(k)$ is large prime but $\#E'(k)$ has small factors.

Example 2: Censor scans network, terminates users who send many elements of $X(E(k))$.

2004 Möller: Fix twist-secure E. Send $X(aP)$ or $X(a'P')$.

Annoying: e.g., consider ECDH.

Same basic issue arises in random-number generation (see, e.g., 2006 Gjøsteen and 2006 Schoenmakers–Sidorenko), password-authenticated key exchange (e.g., 2001 Boyd–Montague–Nguyen, broken 2013), ID-based encryption, etc.

2013 Bernstein–Hamburg–Krasnova–Lange "Elligator: Elliptic-curve points indistinguishable from uniform random strings":

Replace X with fast bijection between large $S \subseteq E(k)$ and, e.g., interval $\{0, 1, \ldots, 2^b - 1\}$.

Alice keeps generating a until $aP \in S$.

Two examples given in paper, both with $\#S \approx 0.5\#E(k)$ for reasonable choices of k.

"Elligator 1",
reinterpreting and simplifying
2013 Fouque–Joux–Tibouchi:

Fix prime power $q \in 3 + 4\mathbf{Z}$;
$s \in \mathbf{F}_q^*$ with
$(s^2 - 2)(s^2 + 2) \neq 0$;
$c = 2/s^2$; $r = c + 1/c$;
$d = -(c+1)^2/(c-1)^2$.

Define $E : x^2 + y^2 = 1 + dx^2y^2$.
This is a complete Edwards curve.

For $\phi : \mathbf{F}_q \to E(\mathbf{F}_q)$ defined
on next slide: the only preimages
of $\phi(t)$ under ϕ are $\{t, -t\}$.

$\phi(\pm 1) = (0, 1)$.

Otherwise $\phi(t) = (x, y)$ where

$u = (1 - t)/(1 + t)$,

$v = u^5 + (r^2 - 2)u^3 + u$,

$X = \chi(v)u$,

$Y =$
$(\chi(v)v)^{(q+1)/4}\chi(v)\chi\left(u^2 + 1/c^2\right)$,

$x = (c - 1)sX(1 + X)/Y$,

$$y = \frac{rX - (1 + X)^2}{rX + (1 + X)^2}.$$

"Elligator 2", 2013 Bernstein–Hamburg–Krasnova–Lange (restricted to the easiest case):

Fix prime power $q \in 1 + 4\mathbf{Z}$;
non-square $u \in \mathbf{F}_q$;
$A, B \in \mathbf{F}_q^*$ with non-square
$A^2 - 4B$; $\sqrt{\ } : \mathbf{F}_q^2 \to \mathbf{F}_q$
with $\sqrt{a^2} \in \{a, -a\}$.

Define $E : y^2 = x^3 + Ax^2 + Bx$.

For $\psi : \mathbf{F}_q \to E(\mathbf{F}_q)$ defined on next slide: the only preimages of $\psi(t)$ under ψ are $\{t, -t\}$.

$\psi(0) = (0, 0)$.

Otherwise $\psi(t) = (x, y)$ where

$v = -A/(1 + ut^2)$,

$\epsilon = \chi(v^3 + Av^2 + Bv)$,

$x = \epsilon v - (1 - \epsilon)A/2$,

$y = -\epsilon\sqrt{x^3 + Ax^2 + Bx}$.

Proofs, inverse maps, etc.:

`elligator.cr.yp.to`

Asymptotic ECDL security

The original ECC advertising:
Index calculus breaks RSA etc.
in subexponential time. Scary!
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Replace XL with Coppersmith
to generalize ECC_2 to ECC?

Concrete ECDL security

Typical for real-world ECC:
the "NIST P-256" curve E :
$y^2 = x^3 - 3x + a_6$ over $\mathbf{F}_q$ where
$q = 2^{256} - 2^{224} + 2^{192} + 2^{96} - 1$,
$a_6 = 41058363725152142129326129780047268409114441015993725554835256314039467401291$.

$E(\mathbf{F}_q)$ has prime order ℓ.
"NIST generator": $P = ($
$48439561293906451759052585252797914202762949526041747995844080717082404635286, \ldots 9)$.

Textbook ECDL cost analysis:

$\approx\sqrt{\pi\ell/2}$ group operations to compute DL in order-ℓ group.

Negation map gains factor $\approx\sqrt{2}$ for elliptic curves.

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But is it the best algorithm that *exists*?

Standard definition of “best”:
minimize “time” (e.g., minimize total “time” over all inputs).

Many researchers have tried and failed to find better ECDL algorithms.

Standard conjecture:
For each $p \in [0, 1]$,
each P-256 ECDL algorithm with success probability $\geq p$ takes “time” $\geq 2^{128} p^{1/2}$.

Interlude regarding "time"

How much "time" does the following algorithm take?

```
def pidigit(n0,n1,n2):
  if n0 == 0:
    if n1 == 0:
      if n2 == 0: return 3
      return           1
    if n2 == 0: return   4
    return               1
  if n1 == 0:
    if n2 == 0: return   5
    return               9
  if n2 == 0: return     2
  return                 6
```

Students in algorithm courses learn to count executed “steps”. Skipped branches take 0 “steps”.

This algorithm uses 4 “steps”.

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Variant: There exists a 258-"step" P-256 discrete-log attack (with 100% success probability). If "time" means "steps" then the standard conjectures are wrong.

1994 Bellare–Kilian–Rogaway:

"*We say that*
A is a (t, q)-adversary if
A runs in at most t steps and
makes at most q queries to $\mathcal{O}$."

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Oops: table-lookup attack
has very small t.

Paper conjectured "useful" DES
security bounds. Any reasonable
interpretation of conjecture was
false, given paper's definition.
Theorems in paper were vacuous.

2000 Bellare–Kilian–Rogaway:
"*We fix some particular Random Access Machine (RAM) as a model of computation. . . . A's running time [means] A's actual execution time plus the length of A's description . . . This convention eliminates pathologies caused [by] arbitrarily large lookup tables . . .*"

2012 Bernstein–Lange:

There are more pathologies!

Assuming plausible heuristics, overwhelmingly verified by computer experiment:

There exists a P-256 ECDL algorithm that takes "time" $\approx 2^{85}$ and has success probability ≈ 1.

"Time" includes algorithm length.

Inescapable conclusion: **The standard conjecture is false.**

Our recommendations to fix the flawed security definitions, conjectures, proofs:

1. Switch from "time" to circuit AT.

(Related, online soon: Improved AT exponents for *batch* NFS.)

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More details and attacks: cr.yp.to/nonuniform.html

DH speed records

Sandy Bridge cycles for high-security constant-time $a, P \mapsto aP$ ("?" if not SUPERCOP-verified):

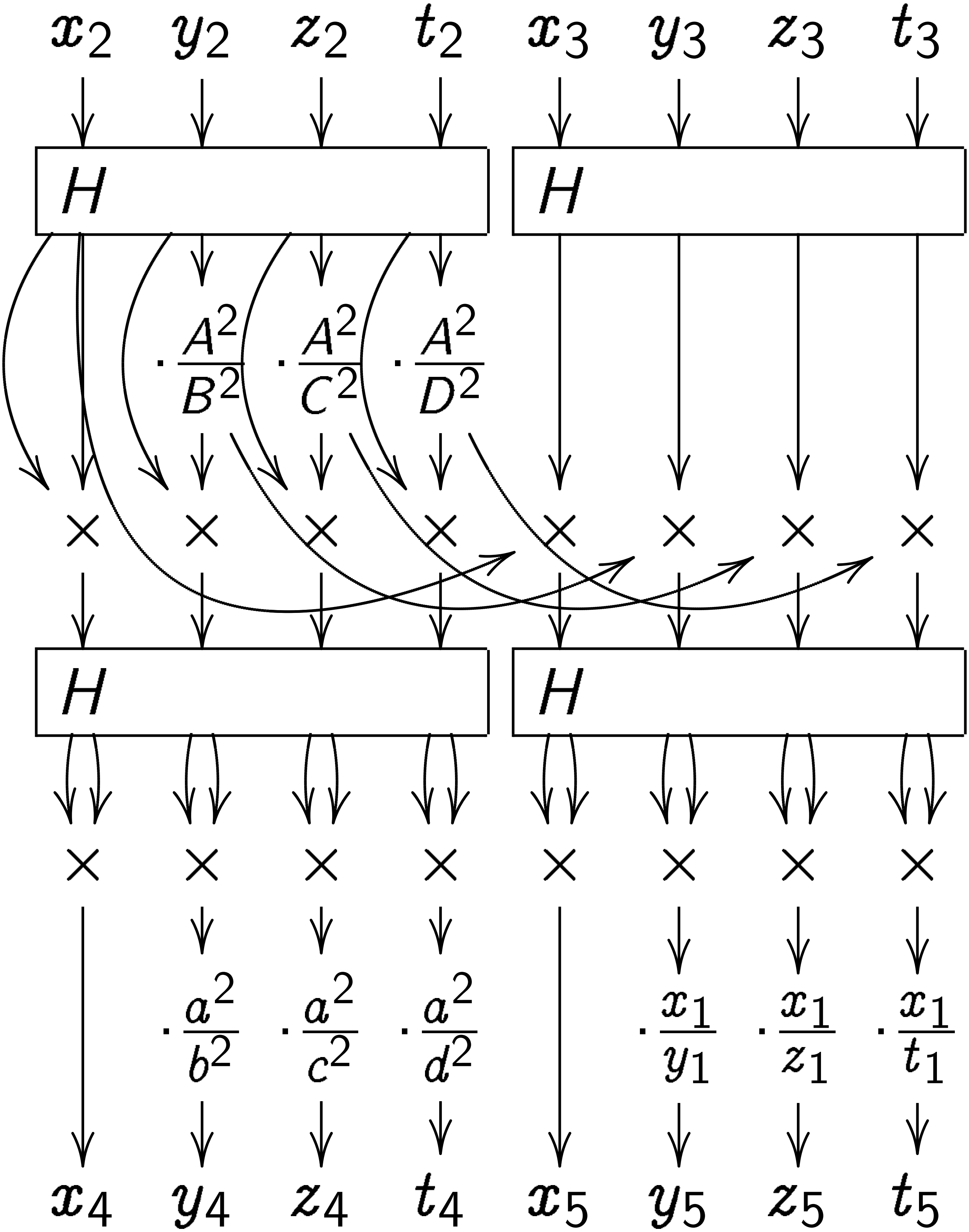
2011 Bernstein–Duif–Lange–Schwabe–Yang: 194120
2012 Hamburg: 153000?
2012 Longa–Sica: 137000?
2013 Bos–Costello–Hisil–Lauter: 122728
2013 Oliveira–López–Aranha–Rodríguez-Henríquez: 114800?
2013 Faz-Hernández–Longa–Sánchez: 96000?
2014 Bernstein–Chuengsatiansup–Lange–Schwabe: 91460

Critical for 122728, 91460:

1986 Chudnovsky–Chudnovsky: traditional Kummer surface allows fast scalar mult. 14**M** for $X(P) \mapsto X(2P)$.

2006 Gaudry: even faster. 25**M** for $X(P), X(Q), X(Q-P)$ $\mapsto X(2P), X(Q+P)$, including 6**M** by surface coefficients.

2012 Gaudry–Schost: 1000000-CPU-hour computation found secure small-coefficient surface over $\mathbf{F}_{2^{127}-1}$.


x_2 y_2 z_2 t_2 x_3 y_3 z_3 t_3
H
H
$\cdot \frac{A^2}{B^2}$ $\cdot \frac{A^2}{C^2}$ $\cdot \frac{A^2}{D^2}$
$\times$ $\times$ $\times$ $\times$ $\times$ $\times$ $\times$ $\times$
H
H
$\times$ $\times$ $\times$ $\times$ $\times$ $\times$ $\times$ $\times$
$\cdot \frac{a^2}{b^2}$ $\cdot \frac{a^2}{c^2}$ $\cdot \frac{a^2}{d^2}$ $\cdot \frac{x_1}{y_1}$ $\cdot \frac{x_1}{z_1}$ $\cdot \frac{x_1}{t_1}$
x_4 y_4 z_4 t_4 x_5 y_5 z_5 t_5

Hyper-and-elliptic-curve crypto

Typical example: Define H :

$$y^2 = (z-1)(z+1)(z+2)$$
$$(z-1/2)(z+3/2)(z-2/3)$$

over $\mathbf{F}_p$ with $p = 2^{127} - 309$;

$J = \text{Jac}\, H$; traditional Kummer surface K; traditional $X : J \to K$.

Small K coeffs $(20 : 1 : 20 : 40)$.

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Warning: There are errors in the Rosenhain/Mumford/Kummer formulas in 2007 Gaudry, 2010 Cosset, 2013 Bos–Costello–Hisil–Lauter. We have simpler, computer-verified formulas.

Define $\mathbf{F}_{p^2} = \mathbf{F}_p[i]/(i^2+1)$;
$r = (7+4i)^2 = 33 + 56i$;
$s = 159 + 56i$; $\omega = \sqrt{-384}$;
$C : y^2 = rx^6 + sx^4 + \overline{s}x^2 + \overline{r}$.

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$$(z, y) \mapsto \left(\frac{1+iz}{1-iz}, \frac{\omega y}{(1-iz)^3}\right)$$
takes H over $\mathbf{F}_{p^2}$ to C.

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Weil restriction W of E, so
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What's new here:
1. Small Kummer coefficients.
Requires lifting Scholten to $\mathbf{Q}$.

2. Explicit formulas for isogenies
$\iota : W \to J$ and $\iota' : J \to W$
with $\iota \circ \iota' = 2$.

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We took random points
in $H(\mathbf{F}_p) \times H(\mathbf{F}_p)$;
applied $H(\mathbf{F}_p) \to C(\mathbf{F}_{p^2})$
$\to E(\mathbf{F}_{p^2}) = W(\mathbf{F}_p)$;
interpolated formulas for ι'.

Similarly interpolated formulas
for ι; verified composition.

Easy computer calculation.
"Wasting brain power
is bad for the environment."

3. Using isogenies to dynamically move computations between $E(\mathbf{F}_{p^2})$ and $J(\mathbf{F}_p)$.

e.g. Generate keys using fast formulas for E.
Compute shared secrets using fast formulas for K.

For more information:
see our talk at ANTS!
Paper coming soon.