High-speed cryptography, part 4: fast multiplication and its applications

Daniel J. Bernstein
University of Illinois at Chicago &
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Survey paper:
cr.yp.to/papers.html#multapps

Integer-factorization bottleneck:
Given sequence of numbers, find nonempty subsequence with square product.
e.g. given $6, 7, 8, 10, 15$, discover $6 \cdot 10 \cdot 15 = 30^2$.

Discrete-log bottleneck:
Given sequence of numbers, find 1 as nontrivial product of powers.
e.g. given $6, 7, 8, 10, 15$, discover $6^3 7^0 8^{-2} 10^3 15^{-3} = 1$.

More generally: find kth power.

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In cryptography, sometimes:
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$\mathbf{C}[x]$-morphism $y \mapsto x^2$ from $\mathbf{C}[x][y]$ to $\mathbf{C}[x]$ maps $f_0(y) + x f_1(y)$ to f.

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Evaluate $f(\alpha)$ for,
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Quickly evaluate $f(\alpha), f(-\alpha$ by evaluating $f_0(\alpha^2)$; $f_1(\alpha^2$

$$f(\alpha) = f_0(\alpha^2) + \alpha f_1(\alpha^2)$$

$$f(-\alpha) = f_0(\alpha^2) - \alpha f_1(\alpha^2)$$

Evaluate $f(\alpha)$ for, e.g., all $\alpha \in \mathbf{C}$ with $\alpha^{1024} = 1$ by evaluating $f_0(\beta)$, $f_1(\beta)$ for all $\beta \in \mathbf{C}$ with $\beta^{512} = 1$; plus 1024 adds, 512 mults.

Apply this recursively $\Rightarrow$ $n \lg n$ adds, $(n/2) \lg n$ mult to evaluate n-coeff f for all $\alpha \in \mathbf{C}$ with $\alpha^n = 1$ if n is a power of 2.

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$c_0 + c_1 x$
f mod x
$(c_0 + c_2$
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$\mathbf{C}[x]$-mo
$\mathbf{C}[x]/(x$
maps c_0
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$_{-1}) \in \mathbf{C}^n$

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$\mathbf{C}[x]$-morphism $\mathbf{C}[$
$\mathbf{C}[x]/(x^2 - 1) \oplus \mathbf{C}$
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If $f \in \mathbf{C}[x]$ and
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$f \bmod x$
$c_0 + c_1 x$
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$(c_0 + \alpha c$
$+ (c_2$
$f \bmod x$
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$+ (c_2$

Given c_0
use n m
$c_0 + \alpha c_n$
$c_0 - \alpha c_n$

$f(\alpha), f(-\alpha)$

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If $f \in \mathbf{C}[x]$ and
$f \bmod x^{2n} - \alpha^2 =$
$c_0 + c_1 x + \cdots + c$
$f \bmod x^n - \alpha =$
$(c_0 + \alpha c_n) + (c_1$
$+ (c_2 + \alpha c_{n+2}$
$f \bmod x^n + \alpha =$
$(c_0 - \alpha c_n) + (c_1$
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Given $c_0, c_1, \ldots, c$
use n mults, $2n$ a
$c_0 + \alpha c_n, c_1 + \alpha c$
$c_0 - \alpha c_n, c_1 - \alpha c$

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$((c_0 + c_2) + (c_1 + c_3)x,$
$(c_0 - c_2) + (c_1 - c_3)x)$.

If $f \in \mathbf{C}[x]$ and
$f \bmod x^{2n} - \alpha^2 =$
$c_0 + c_1 x + \cdots + c_{2n-1} x^{2n-}$
$f \bmod x^n - \alpha =$
$(c_0 + \alpha c_n) + (c_1 + \alpha c_{n+1})x$
$+ (c_2 + \alpha c_{n+2})x^2 + \cdots,$
$f \bmod x^n + \alpha =$
$(c_0 - \alpha c_n) + (c_1 - \alpha c_{n+1})x$
$+ (c_2 - \alpha c_{n+2})x^2 + \cdots.$

Given $c_0, c_1, \ldots, c_{2n-1} \in \mathbf{C}$,
use n mults, $2n$ adds to cor
$c_0 + \alpha c_n, c_1 + \alpha c_{n+1}, \ldots,$
$c_0 - \alpha c_n, c_1 - \alpha c_{n+1}, \ldots.$

Another view of the FFT

If $f \in \mathbf{C}[x]$ and
$f \bmod x^4 - 1 =$
$c_0 + c_1 x + c_2 x^2 + c_3 x^3$ then
$f \bmod x^2 - 1 =$
$(c_0 + c_2) + (c_1 + c_3)x$,
$f \bmod x^2 + 1 =$
$(c_0 - c_2) + (c_1 - c_3)x$.

$\mathbf{C}[x]$-morphism $\mathbf{C}[x]/(x^4 - 1) \hookrightarrow\!\!\!\!\rightarrow$
$\mathbf{C}[x]/(x^2 - 1) \oplus \mathbf{C}[x]/(x^2 + 1)$
maps $c_0 + c_1 x + c_2 x^2 + c_3 x^3$ to
$((c_0 + c_2) + (c_1 + c_3)x,$
$(c_0 - c_2) + (c_1 - c_3)x)$.

If $f \in \mathbf{C}[x]$ and
$f \bmod x^{2n} - \alpha^2 =$
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$f \bmod x^n - \alpha =$
$(c_0 + \alpha c_n) + (c_1 + \alpha c_{n+1})x$
$\quad + (c_2 + \alpha c_{n+2})x^2 + \cdots$,
$f \bmod x^n + \alpha =$
$(c_0 - \alpha c_n) + (c_1 - \alpha c_{n+1})x$
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$c_0 - \alpha c_n, c_1 - \alpha c_{n+1}, \ldots.$

view of the FFT

$[x]$ and

$^4 - 1 =$

$+ c_2x^2 + c_3x^3$ then

$^2 - 1 =$

$) + (c_1 + c_3)x,$

$^2 + 1 =$

$) + (c_1 - c_3)x.$

rphism $\mathbf{C}[x]/(x^4 - 1) \hookrightarrow\!\!\!\!\rightarrow$

$^2 - 1) \oplus \mathbf{C}[x]/(x^2 + 1)$

$+ c_1x + c_2x^2 + c_3x^3$ to

$_2) + (c_1 + c_3)x,$

$_2) + (c_1 - c_3)x).$

If $f \in \mathbf{C}[x]$ and

$f \bmod x^{2n} - \alpha^2 =$

$c_0 + c_1x + \cdots + c_{2n-1}x^{2n-1}$ then

$f \bmod x^n - \alpha =$

$(c_0 + \alpha c_n) + (c_1 + \alpha c_{n+1})x$

$\quad + (c_2 + \alpha c_{n+2})x^2 + \cdots,$

$f \bmod x^n + \alpha =$

$(c_0 - \alpha c_n) + (c_1 - \alpha c_{n+1})x$

$\quad + (c_2 - \alpha c_{n+2})x^2 + \cdots.$

Given $c_0, c_1, \ldots, c_{2n-1} \in \mathbf{C}$,

use n mults, $2n$ adds to compute

$c_0 + \alpha c_n, c_1 + \alpha c_{n+1}, \ldots,$

$c_0 - \alpha c_n, c_1 - \alpha c_{n+1}, \ldots.$

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c_3x^3 then

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$c_3)x$.

$x]/(x^4-1) \hookrightarrow\!\!\!\!\twoheadrightarrow$

$\mathbf{C}[x]/(x^2+1)$

$c_2x^2 + c_3x^3$ to

$c_3)x$,

$c_3)x)$.

If $f \in \mathbf{C}[x]$ and
$f \bmod x^{2n} - \alpha^2 =$
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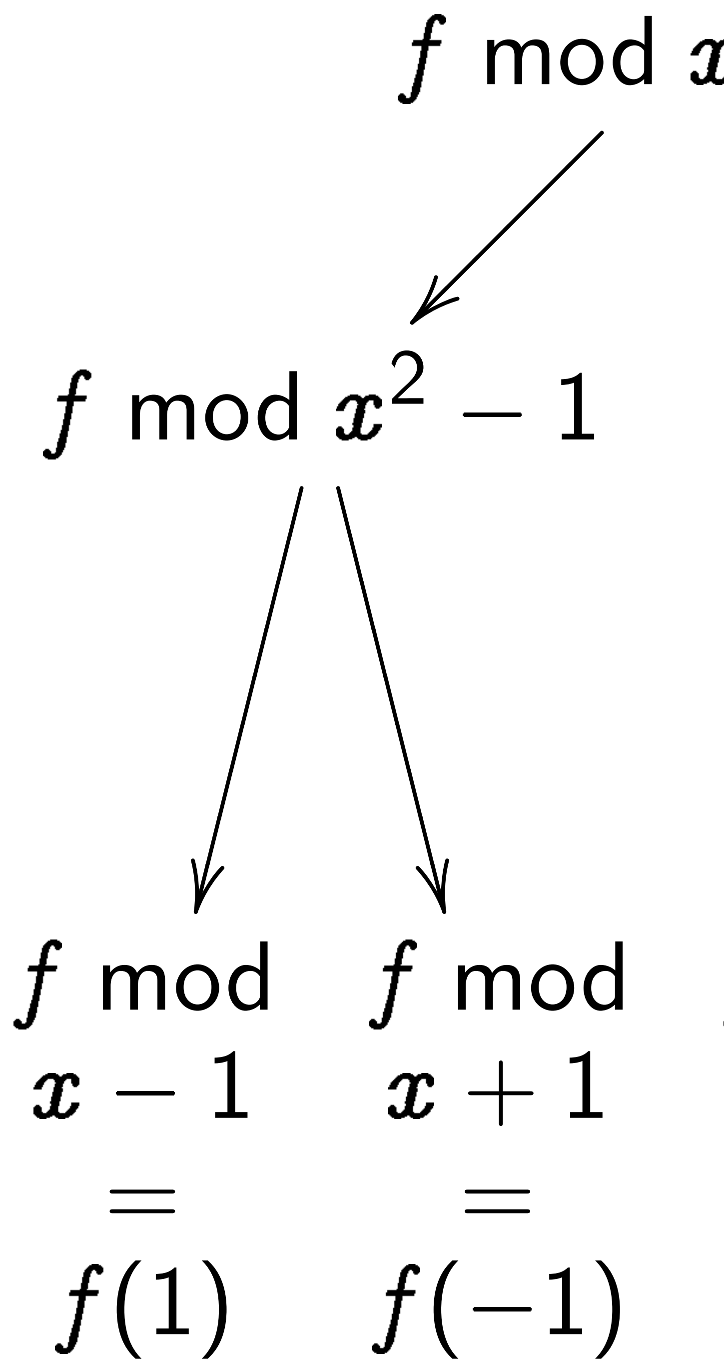
Given $c_0, c_1, \ldots, c_{2n-1} \in \mathbf{C}$,
use n mults, $2n$ adds to compute
$c_0 + \alpha c_n, c_1 + \alpha c_{n+1}, \ldots,$
$c_0 - \alpha c_n, c_1 - \alpha c_{n+1}, \ldots.$

Apply this recursiv



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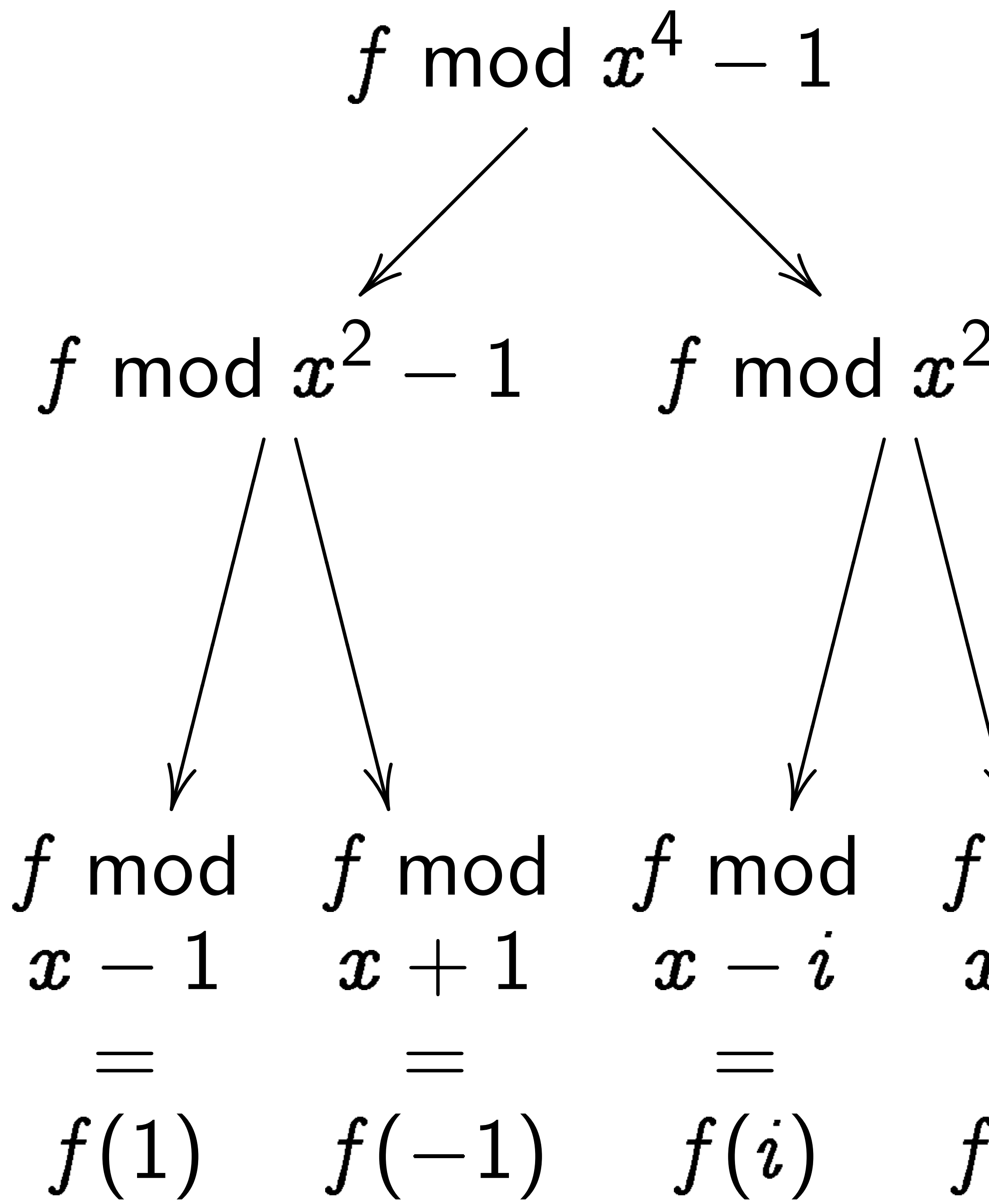
If $f \in \mathbf{C}[x]$ and
$f \bmod x^{2n} - \alpha^2 =$
$c_0 + c_1 x + \cdots + c_{2n-1} x^{2n-1}$ then
$f \bmod x^n - \alpha =$
$(c_0 + \alpha c_n) + (c_1 + \alpha c_{n+1})x$
$\quad + (c_2 + \alpha c_{n+2})x^2 + \cdots,$
$f \bmod x^n + \alpha =$
$(c_0 - \alpha c_n) + (c_1 - \alpha c_{n+1})x$
$\quad + (c_2 - \alpha c_{n+2})x^2 + \cdots.$

Given $c_0, c_1, \ldots, c_{2n-1} \in \mathbf{C}$,
use n mults, $2n$ adds to compute
$c_0 + \alpha c_n, c_1 + \alpha c_{n+1}, \ldots,$
$c_0 - \alpha c_n, c_1 - \alpha c_{n+1}, \ldots.$

Apply this recursively:



(basic FFT idea: 1866 Gaus
this view: 1972 Fiduccia)

If $f \in \mathbf{C}[x]$ and

$f \bmod x^{2n} - \alpha^2 =$

$c_0 + c_1 x + \cdots + c_{2n-1} x^{2n-1}$ then

$f \bmod x^n - \alpha =$

$$(c_0 + \alpha c_n) + (c_1 + \alpha c_{n+1})x + (c_2 + \alpha c_{n+2})x^2 + \cdots,$$

$f \bmod x^n + \alpha =$

$$(c_0 - \alpha c_n) + (c_1 - \alpha c_{n+1})x + (c_2 - \alpha c_{n+2})x^2 + \cdots.$$

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use n mults, $2n$ adds to compute

$c_0 + \alpha c_n, c_1 + \alpha c_{n+1}, \ldots,$

$c_0 - \alpha c_n, c_1 - \alpha c_{n+1}, \ldots.$

Apply this recursively:

$f \bmod x^4 - 1$

$f \bmod x^2 - 1$ $\qquad$ $f \bmod x^2 + 1$

$f \bmod x - 1 = f(1)$	$f \bmod x + 1 = f(-1)$	$f \bmod x - i = f(i)$	$f \bmod x + i = f(-i)$

(basic FFT idea: 1866 Gauss;

this view: 1972 Fiduccia)

$[x]$ and

$^{2n} - \alpha^2 =$

$+ \cdots + c_{2n-1}x^{2n-1}$ then

$^{n} - \alpha =$

$_n) + (c_1 + \alpha c_{n+1})x$

$+ \alpha c_{n+2})x^2 + \cdots,$

$^{n} + \alpha =$

$_n) + (c_1 - \alpha c_{n+1})x$

$- \alpha c_{n+2})x^2 + \cdots.$

$, c_1, \ldots, c_{2n-1} \in \mathbf{C},$

ults, $2n$ adds to compute

$_n, c_1 + \alpha c_{n+1}, \ldots,$

$_n, c_1 - \alpha c_{n+1}, \ldots.$

Apply this recursively:

$f \bmod x^4 - 1$

$f \bmod x^2 - 1$ $\quad$ $f \bmod x^2 + 1$

f mod $x - 1$ $=$ $f(1)$	f mod $x + 1$ $=$ $f(-1)$	f mod $x - i$ $=$ $f(i)$	f mod $x + i$ $=$ $f(-i)$

(basic FFT idea: 1866 Gauss;
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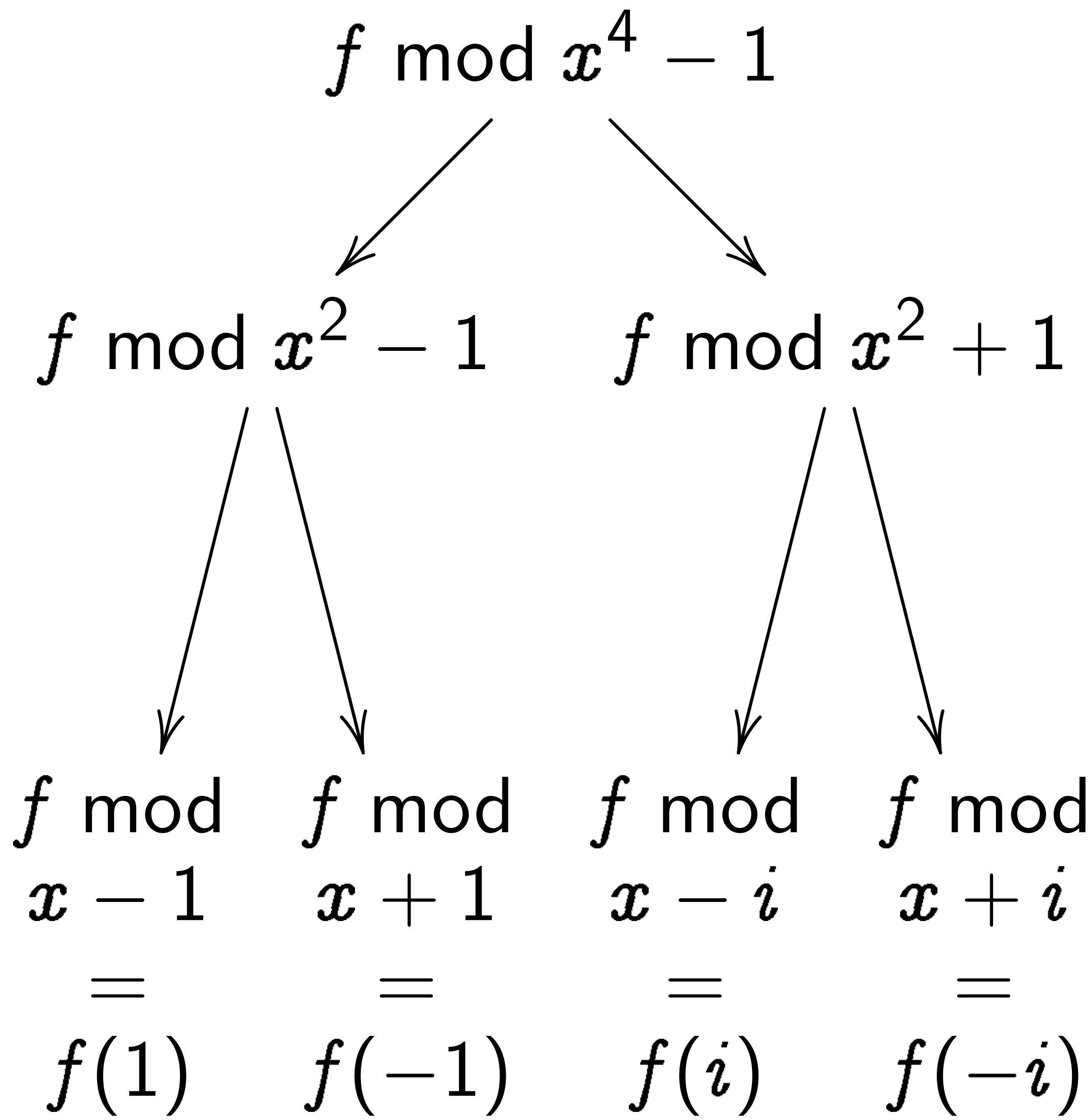
$_{n+1}, \ldots.$

Apply this recursively:



(basic FFT idea: 1866 Gauss; this view: 1972 Fiduccia)

1966 Sande, 1966

Can very quickly m

in $\mathbf{C}[x]/(x^n - 1)$ c

by mapping $\mathbf{C}[x]/$

Given $f, g \in \mathbf{C}[x]/$

compute fg as T^{-}

using $T : \mathbf{C}[x]/(x^n$

Compute T quickl

Given $f, g \in \mathbf{C}[x]$,

compute fg from

its image in $\mathbf{C}[x]/$

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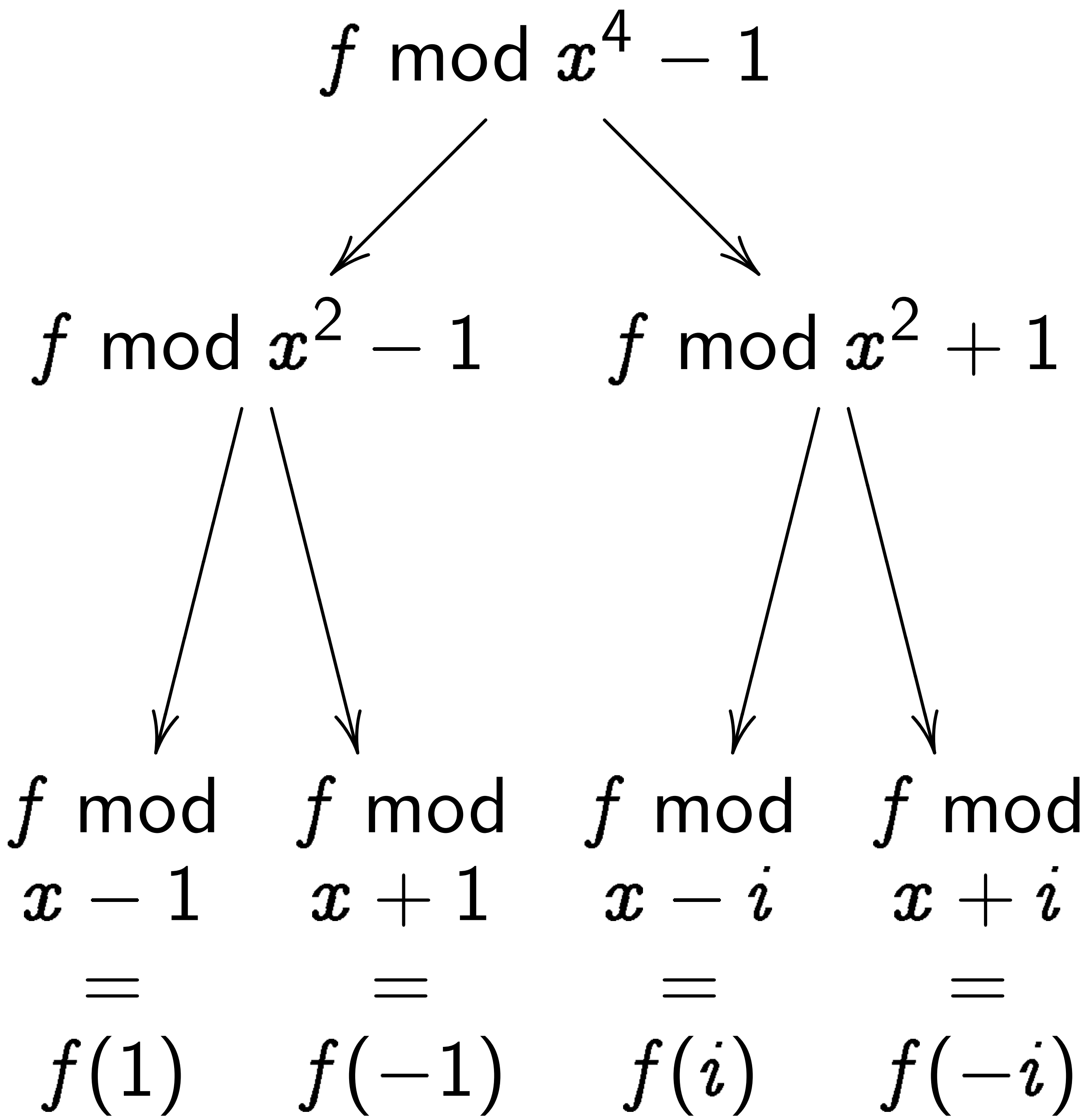
Apply this recursively:



(basic FFT idea: 1866 Gauss;
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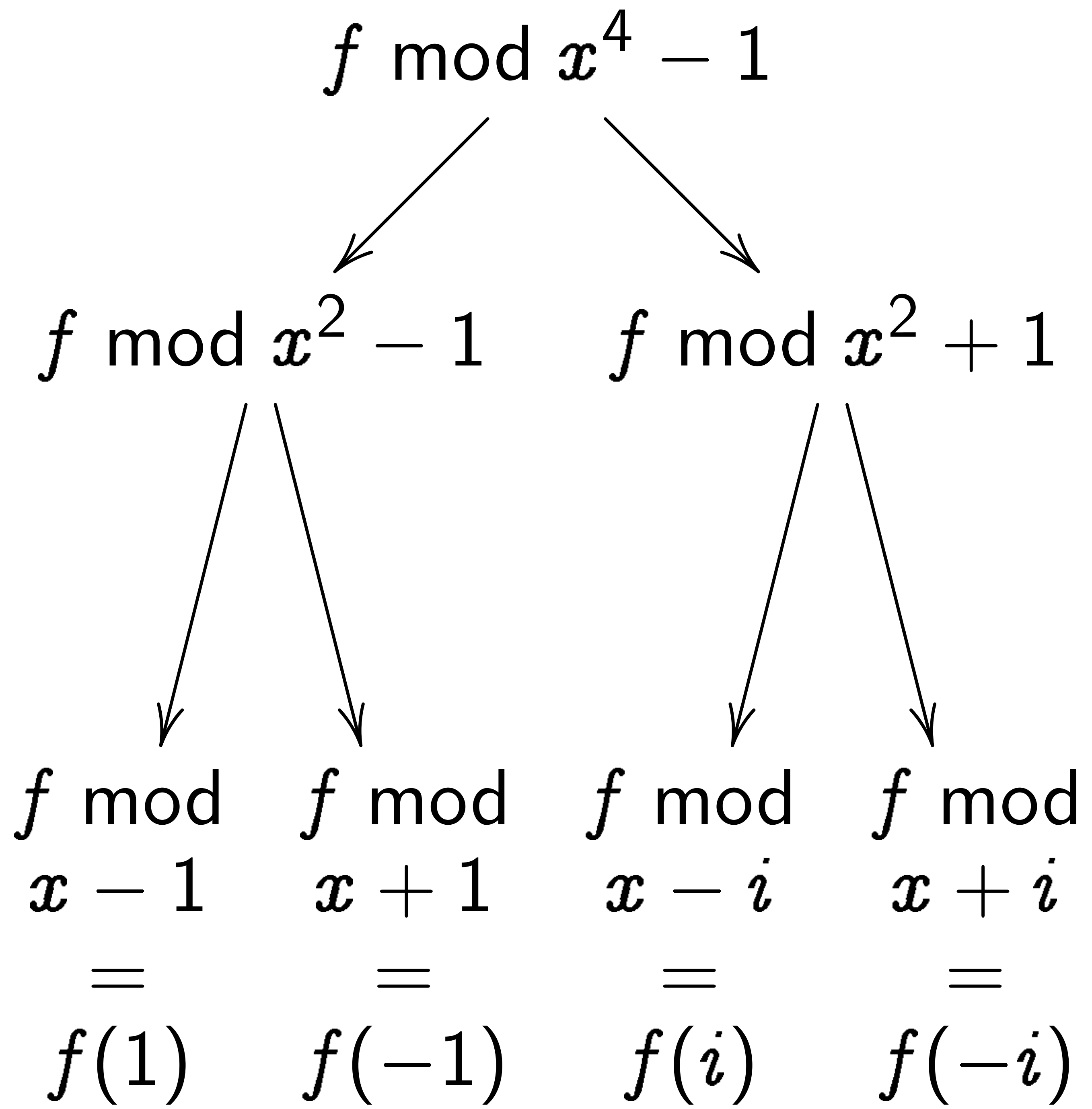
1966 Sande, 1966 Stockham
Can very quickly multiply
in $\mathbf{C}[x]/(x^n - 1)$ or $\mathbf{C}[x]$ or
by mapping $\mathbf{C}[x]/(x^n - 1)$ t

Given $f, g \in \mathbf{C}[x]/(x^n - 1)$:
compute fg as $T^{-1}(T(f)T($
using $T : \mathbf{C}[x]/(x^n - 1) \hookrightarrow\!\!\!\!\rightarrow$
Compute T quickly by the F

Given $f, g \in \mathbf{C}[x]$, $\deg fg <$
compute fg from
its image in $\mathbf{C}[x]/(x^n - 1)$.

Apply this recursively:



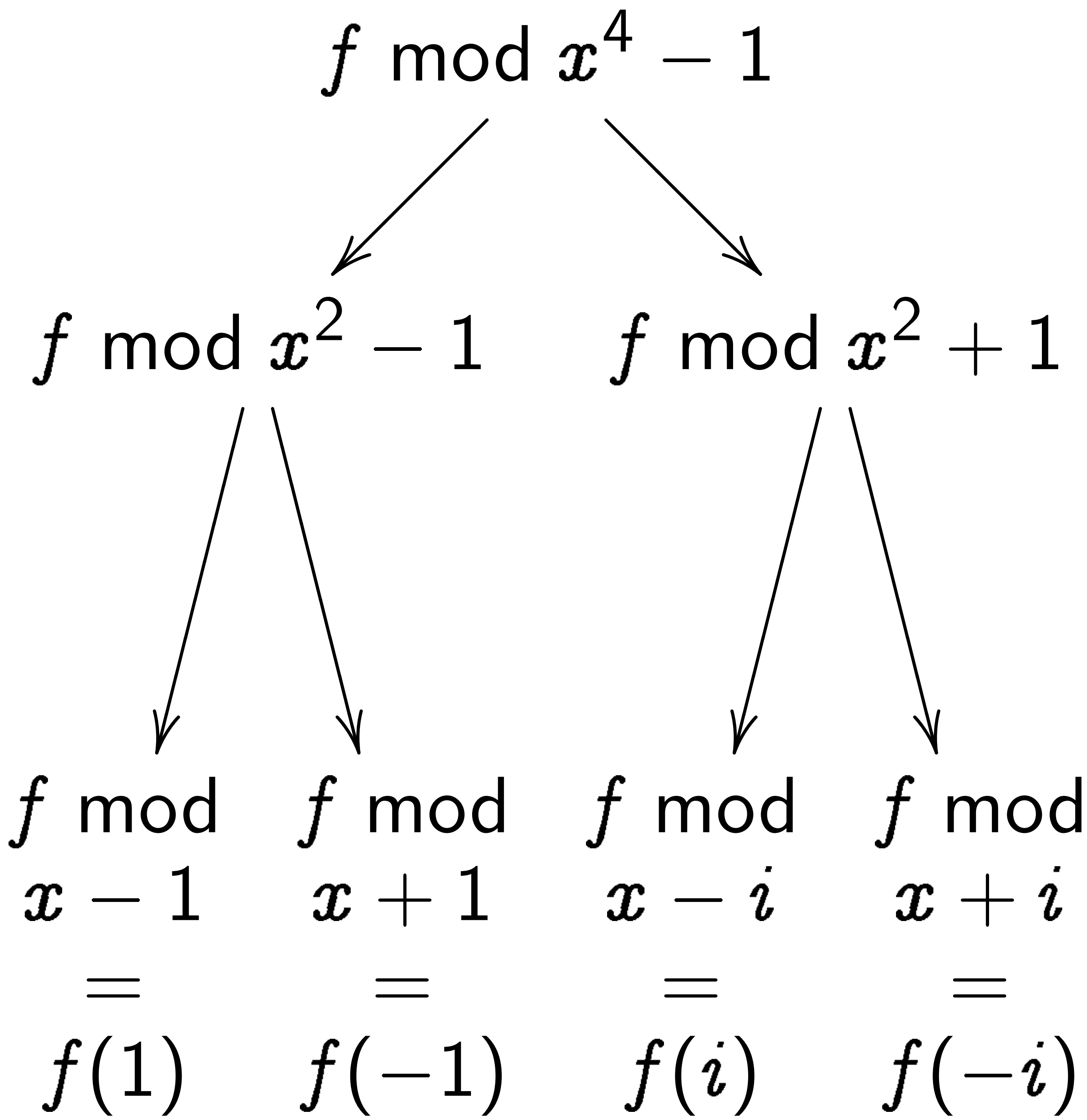
(basic FFT idea: 1866 Gauss; this view: 1972 Fiduccia)

1966 Sande, 1966 Stockham: Can very quickly multiply in $\mathbf{C}[x]/(x^n - 1)$ or $\mathbf{C}[x]$ or $\mathbf{R}[x]$ by mapping $\mathbf{C}[x]/(x^n - 1)$ to $\mathbf{C}^n$.

Given $f, g \in \mathbf{C}[x]/(x^n - 1)$: compute fg as $T^{-1}(T(f)T(g))$ using $T : \mathbf{C}[x]/(x^n - 1) \hookrightarrow\!\!\!\rightarrow \mathbf{C}^n$. Compute T quickly by the FFT.

Given $f, g \in \mathbf{C}[x]$, $\deg fg < n$: compute fg from its image in $\mathbf{C}[x]/(x^n - 1)$.

Apply this recursively:



(basic FFT idea: 1866 Gauss;
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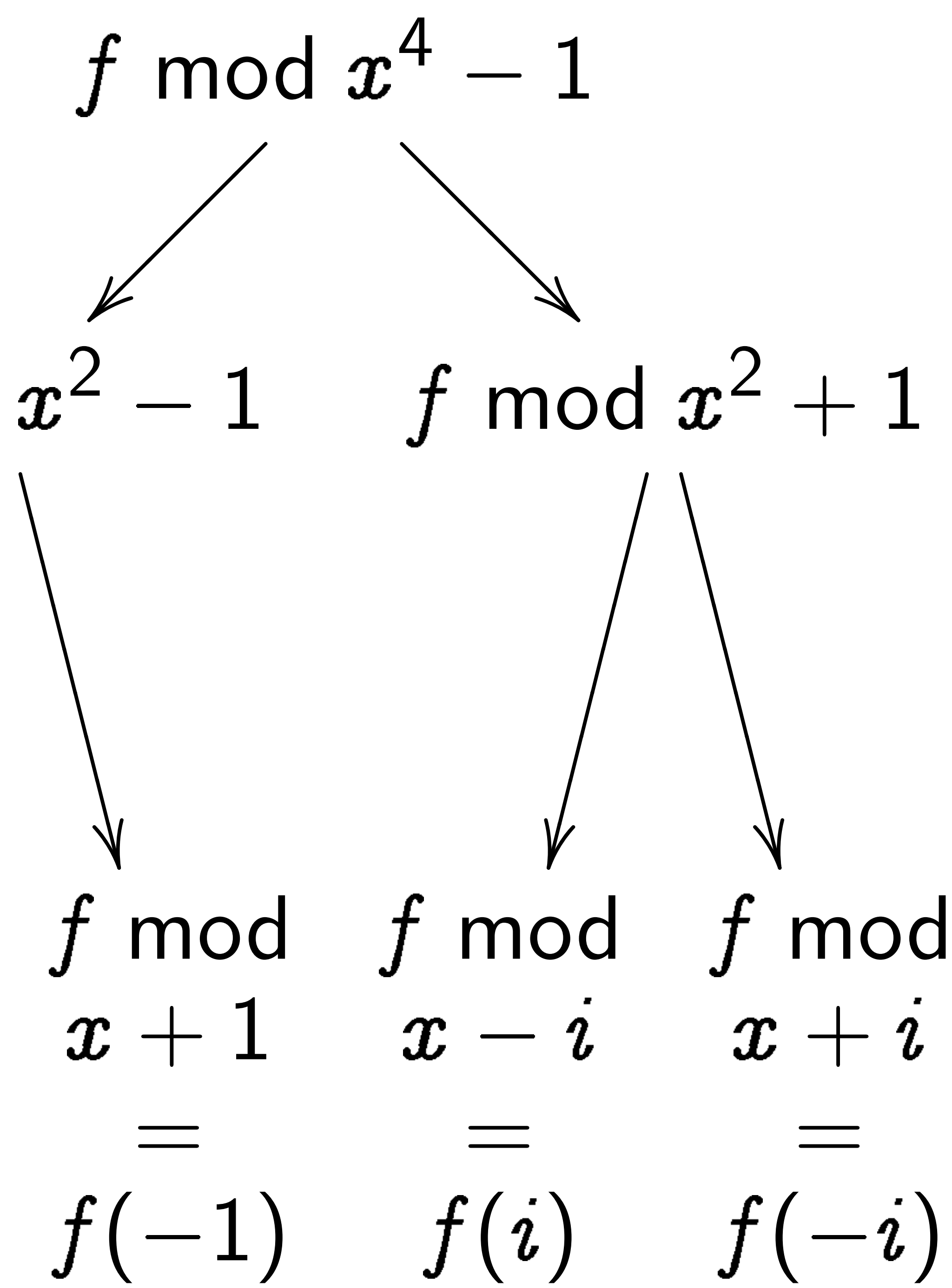
Given $f, g \in \mathbf{C}[x]$, $\deg fg < n$:
compute fg from
its image in $\mathbf{C}[x]/(x^n - 1)$.

Later authors: Replace $\mathbf{C}$ with,
e.g., $R = \mathbf{Z}/(3 \cdot 2^{41} + 1)$;
23 has order 2^{41} in R^*.

his recursively:



FT idea: 1866 Gauss;
v: 1972 Fiduccia)

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$f \bmod x^2 + 1$

$f \bmod x - i = f(i)$ $\quad$ $f \bmod x + i = f(-i)$

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Multiplication and

Given $r, s \in \mathbf{Z}$, ca
in time $\leq b(\lg b)^{1+}$
where b is number

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Also time $\leq b(\lg b$
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Given $r, s \in \mathbf{Z}$ wit
compute $\lfloor r/s \rfloor$ an

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1966 Sande, 1966 Stockham:
Can very quickly multiply
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Multiplication and division

Given $r, s \in \mathbf{Z}$, can compute
in time $\le b(\lg b)^{1+o(1)}$
where b is number of input b

(1971 Pollard; independently
1971 Nicholson; independent
1971 Schönhage Strassen)

Also time $\le b(\lg b)^{1+o(1)}$
where b is number of input b
Given $r, s \in \mathbf{Z}$ with $s \ne 0$,
compute $\lfloor r/s \rfloor$ and $r \bmod s$

(reduction to product:
1966 Cook)

1966 Sande, 1966 Stockham:
Can very quickly multiply
in $\mathbf{C}[x]/(x^n-1)$ or $\mathbf{C}[x]$ or $\mathbf{R}[x]$
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<u>Multiplication and division</u>

Given $r, s \in \mathbf{Z}$, can compute rs
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e T quickly by the FFT.

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(reduction to product: 1966 Cook)

Product

Time $\le$

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or $\mathbf{C}[x]$ or $\mathbf{R}[x]$

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$\deg fg < n$:

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Multiplication and division

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Also time $\le b(\lg b)^{1+o(1)}$ where b is number of input bits: Given $r, s \in \mathbf{Z}$ with $s \neq 0$, compute $\lfloor r/s \rfloor$ and $r \bmod s$.

(reduction to product: 1966 Cook)

Product trees

Time $\le b(\lg b)^{2+o}$

where b is number

Given $x_1, x_2, \ldots,$

compute $x_1 x_2 \cdots$

Actually compute

product tree of x

Root is $x_1 x_2 \cdots x$

Has left subtree if

product tree of x_1

Also right subtree

product tree of $x_{\lceil}$

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$\mathbf{R}[x]$
to $\mathbf{C}^n$.

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Multiplication and division

Given $r, s \in \mathbf{Z}$, can compute rs
in time $\le b(\lg b)^{1+o(1)}$
where b is number of input bits.

(1971 Pollard; independently
1971 Nicholson; independently
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Also time $\le b(\lg b)^{1+o(1)}$
where b is number of input bits:
Given $r, s \in \mathbf{Z}$ with $s \ne 0$,
compute $\lfloor r/s \rfloor$ and $r \bmod s$.

(reduction to product:
1966 Cook)

Product trees

Time $\le b(\lg b)^{2+o(1)}$
where b is number of input
Given $x_1, x_2, \ldots, x_n \in \mathbf{Z}$,
compute $x_1 x_2 \cdots x_n$.

Actually compute
product tree of $x_1, x_2, \ldots,$
Root is $x_1 x_2 \cdots x_n$.
Has left subtree if $n \ge 2$:
product tree of $x_1, \ldots, x_{\lceil n/}$
Also right subtree if $n \ge 2$:
product tree of $x_{\lceil n/2 \rceil + 1}, \ldots$

Multiplication and division

Given $r, s \in \mathbf{Z}$, can compute rs in time $\leq b(\lg b)^{1+o(1)}$ where b is number of input bits.

(1971 Pollard; independently 1971 Nicholson; independently 1971 Schönhage Strassen)

Also time $\leq b(\lg b)^{1+o(1)}$ where b is number of input bits: Given $r, s \in \mathbf{Z}$ with $s \neq 0$, compute $\lfloor r/s \rfloor$ and $r \bmod s$.

(reduction to product: 1966 Cook)

Product trees

Time $\leq b(\lg b)^{2+o(1)}$ where b is number of input bits: Given $x_1, x_2, \ldots, x_n \in \mathbf{Z}$, compute $x_1 x_2 \cdots x_n$.

Actually compute **product tree** of $x_1, x_2, \ldots, x_n$. Root is $x_1 x_2 \cdots x_n$. Has left subtree if $n \geq 2$: product tree of $x_1, \ldots, x_{\lceil n/2 \rceil}$. Also right subtree if $n \geq 2$: product tree of $x_{\lceil n/2 \rceil + 1}, \ldots, x_n$.

cation and division

$s \in \mathbf{Z}$, can compute rs
$\leq b(\lg b)^{1+o(1)}$
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ne $\leq b(\lg b)^{1+o(1)}$
is number of input bits:
$s \in \mathbf{Z}$ with $s \neq 0$,
e $\lfloor r/s \rfloor$ and $r \bmod s$.

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Product trees

Time $\leq b(\lg b)^{2+o(1)}$
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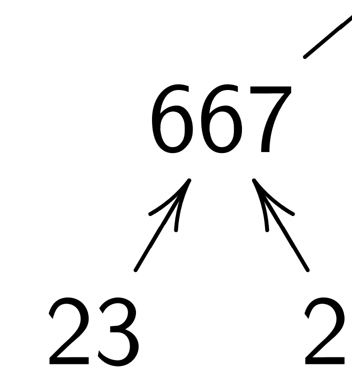
Given $x_1, x_2, \ldots, x_n \in \mathbf{Z}$,
compute $x_1 x_2 \cdots x_n$.

Actually compute
product tree of $x_1, x_2, \ldots, x_n$.
Root is $x_1 x_2 \cdots x_n$.
Has left subtree if $n \geq 2$:
product tree of $x_1, \ldots, x_{\lceil n/2 \rceil}$.
Also right subtree if $n \geq 2$:
product tree of $x_{\lceil n/2 \rceil + 1}, \ldots, x_n$.

e.g. tree



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r of input bits:
h $s \neq 0$,
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Product trees

Time $\le b(\lg b)^{2+o(1)}$
where b is number of input bits:
Given $x_1, x_2, \ldots, x_n \in \mathbf{Z}$,
compute $x_1 x_2 \cdots x_n$.

Actually compute
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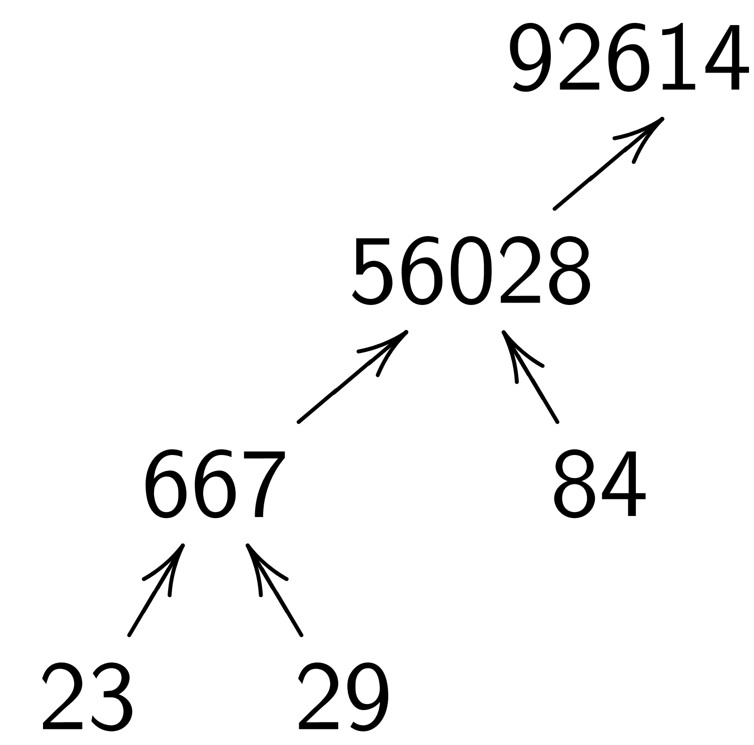
e.g. tree for 23, 29



Tree has $\le (\lg b)^{1+}$
Each level has $\le b$

Obtain each level
in time $\le b(\lg b)^{1+}$
by multiplying low

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Product trees

Time $\leq b(\lg b)^{2+o(1)}$
where b is number of input bits:
Given $x_1, x_2, \ldots, x_n \in \mathbf{Z}$,
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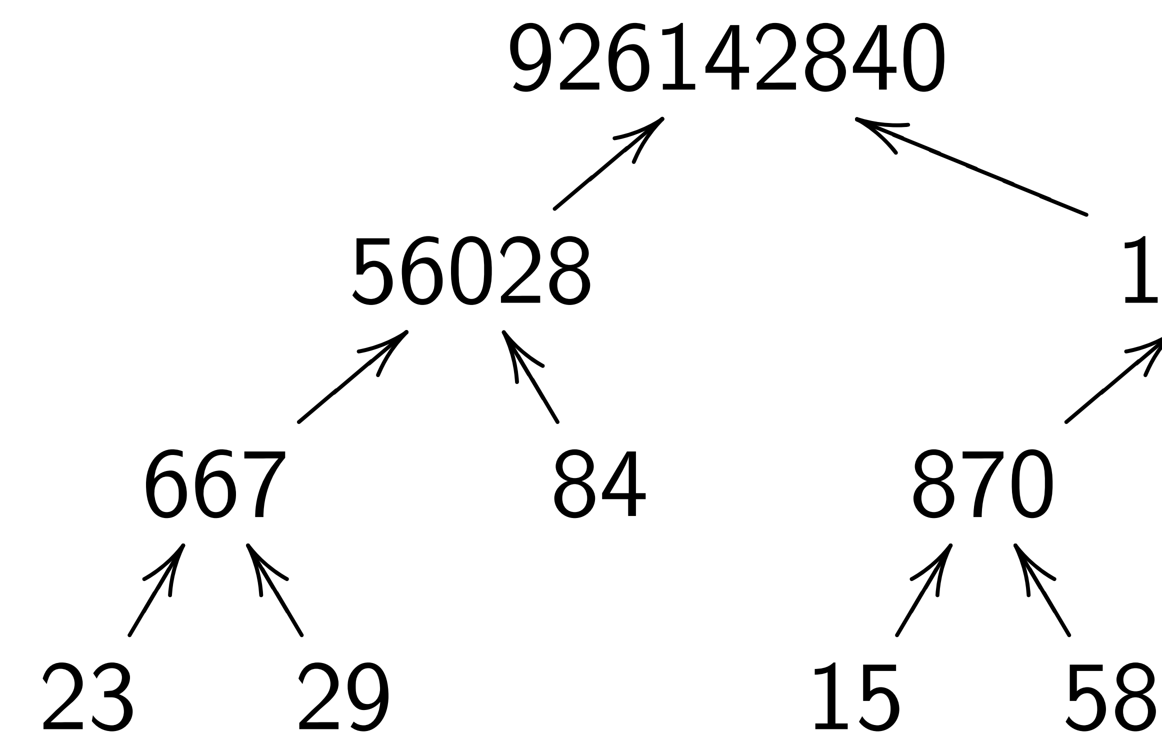
e.g. tree for 23, 29, 84, 15, 58



Tree has $\leq (\lg b)^{1+o(1)}$ level
Each level has $\leq b(\lg b)^{0+o(1}$

Obtain each level
in time $\leq b(\lg b)^{1+o(1)}$
by multiplying lower-level pa

Product trees

Time $\leq b(\lg b)^{2+o(1)}$
where b is number of input bits:
Given $x_1, x_2, \ldots, x_n \in \mathbf{Z}$,
compute $x_1 x_2 \cdots x_n$.

Actually compute
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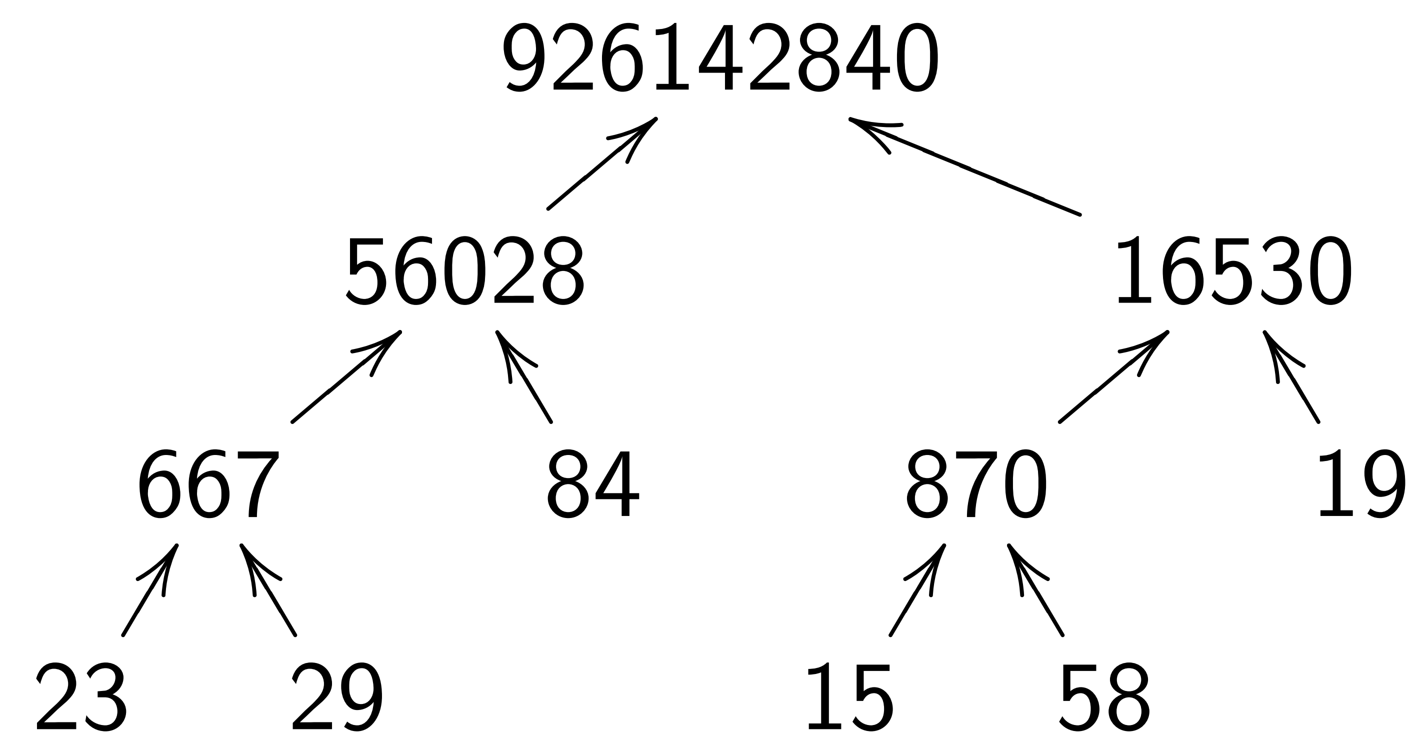
e.g. tree for $23, 29, 84, 15, 58, 19$:



Tree has $\leq (\lg b)^{1+o(1)}$ levels.
Each level has $\leq b(\lg b)^{0+o(1)}$ bits.

Obtain each level
in time $\leq b(\lg b)^{1+o(1)}$
by multiplying lower-level pairs.

trees

$b(\lg b)^{2+o(1)}$

is number of input bits:

$_1, x_2, \ldots, x_n \in \mathbf{Z}$,

e $x_1 x_2 \cdots x_n$.

compute

tree of $x_1, x_2, \ldots, x_n$.

$x_1 x_2 \cdots x_n$.

subtree if $n \geq 2$:

tree of $x_1, \ldots, x_{\lceil n/2 \rceil}$.

nt subtree if $n \geq 2$:

tree of $x_{\lceil n/2 \rceil + 1}, \ldots, x_n$.

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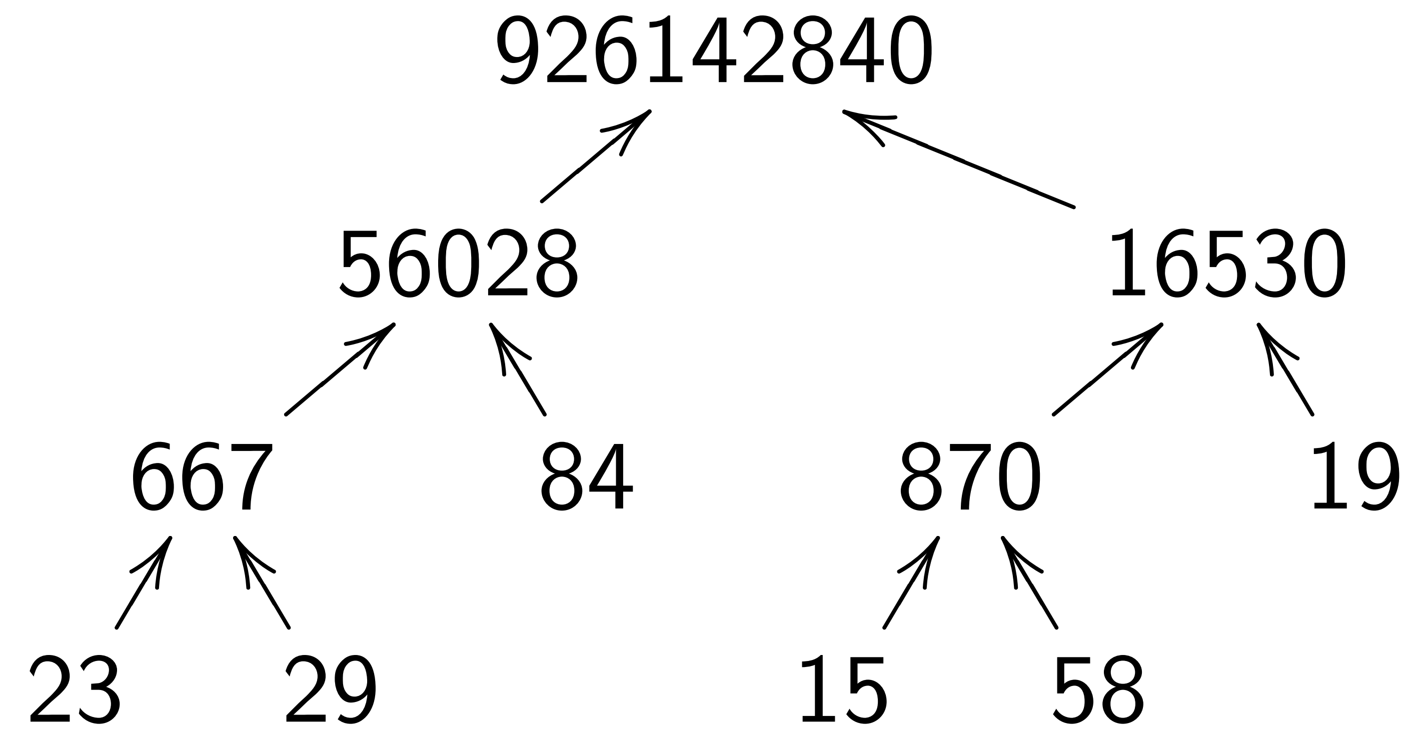

Tree has $\leq (\lg b)^{1+o(1)}$ levels.

Each level has $\leq b(\lg b)^{0+o(1)}$ bits.

Obtain each level

in time $\leq b(\lg b)^{1+o(1)}$

by multiplying lower-level pairs.

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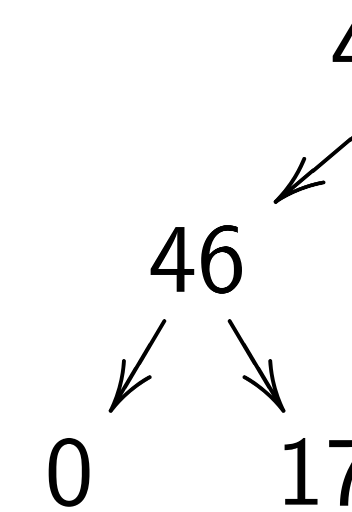

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r of input bits:

$x_n \in \mathbf{Z}$,

x_n.

$_1, x_2, \ldots, x_n$.

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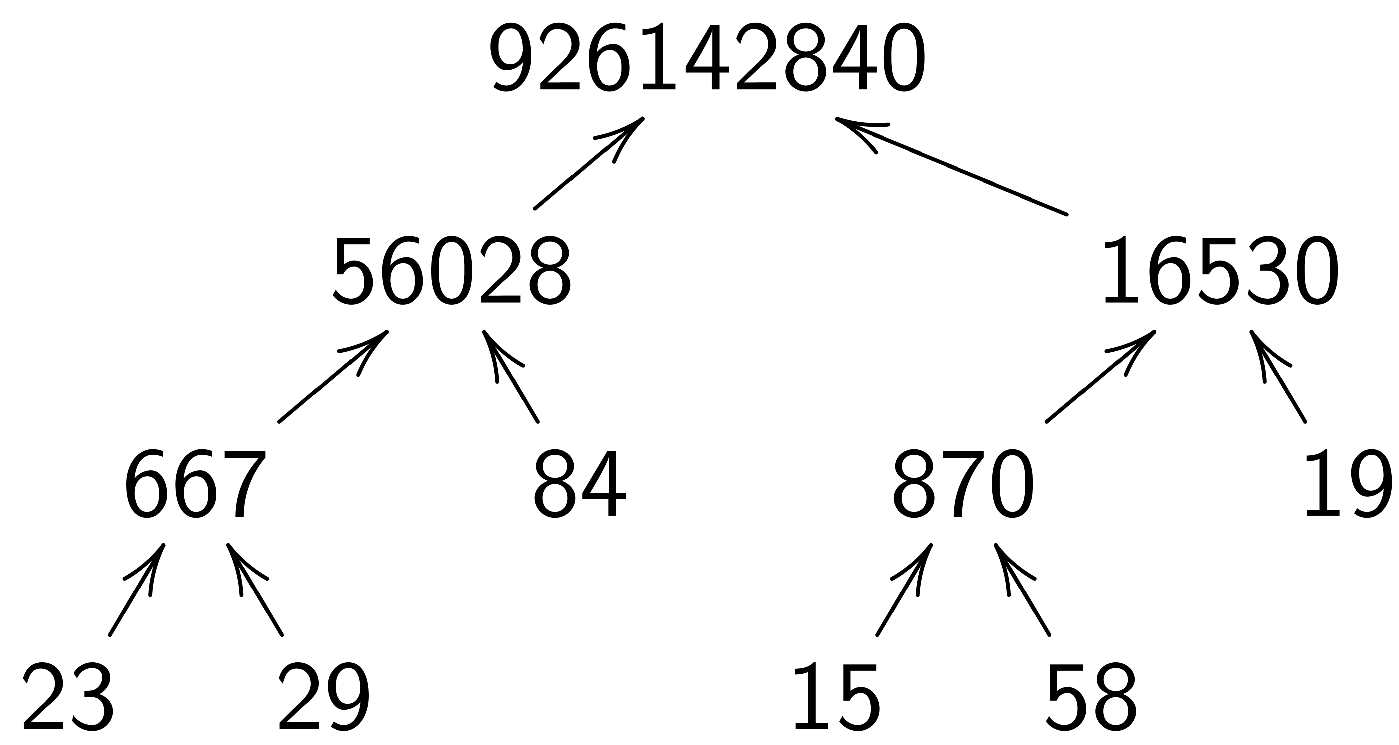
$n \geq 2$:

$, \ldots, x_{\lceil n/2 \rceil}$.

if $n \geq 2$:

$n/2\rceil+1, \ldots, x_n$.

e.g. tree for 23, 29, 84, 15, 58, 19:



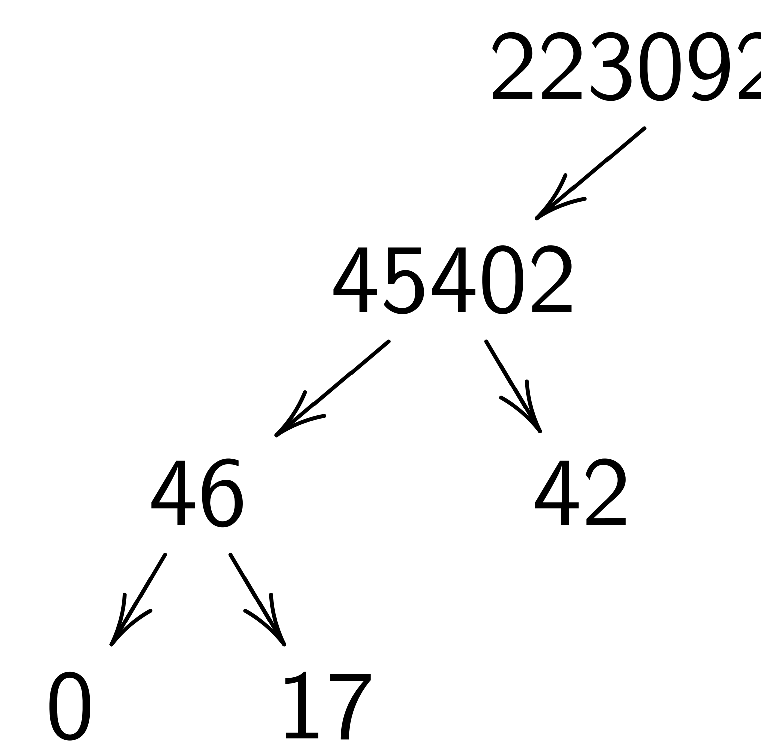
Tree has $\leq (\lg b)^{1+o(1)}$ levels.
Each level has $\leq b(\lg b)^{0+o(1)}$ bits.

Obtain each level
in time $\leq b(\lg b)^{1+o(1)}$
by multiplying lower-level pairs.

Remainder trees

Remainder tree
of $r, x_1, x_2, \ldots, x_r$
one node $r \bmod t$
in product tree of

e.g. remainder tree
223092870, 23, 29,



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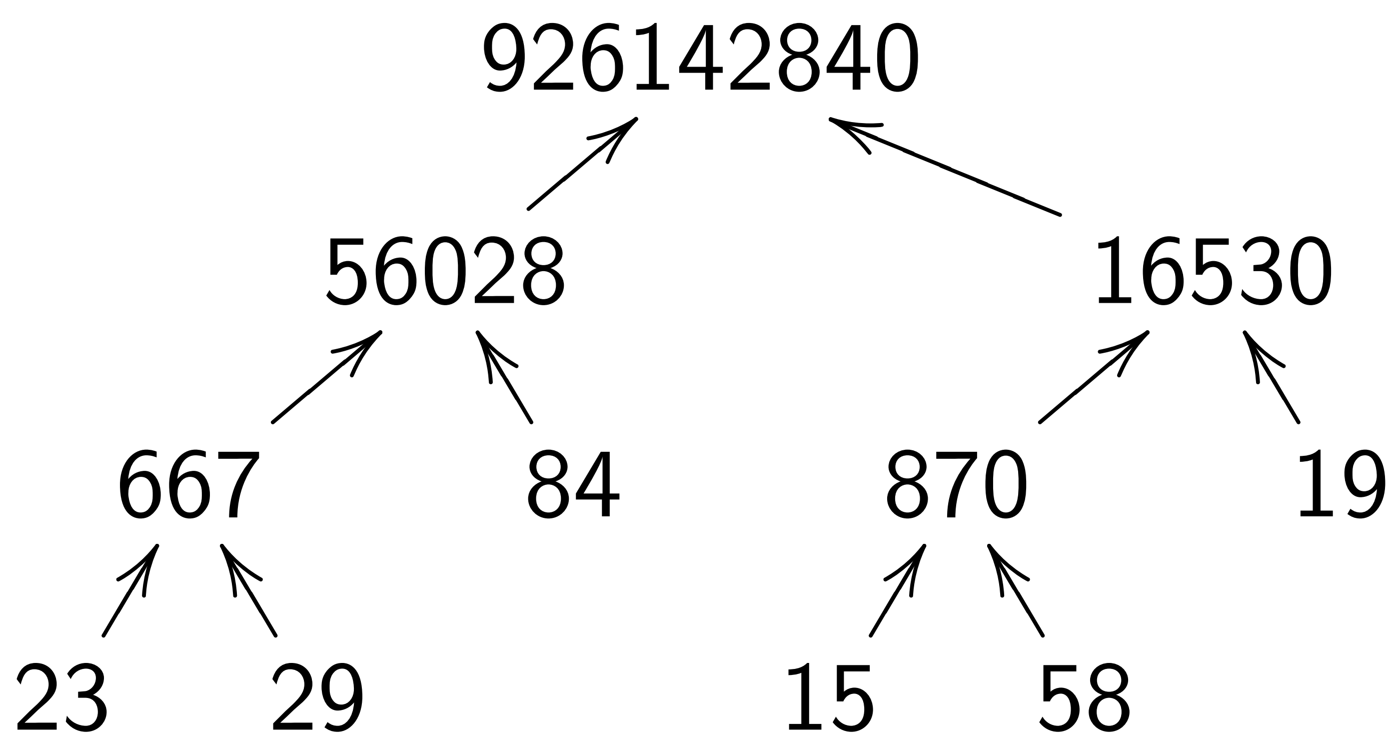
e.g. tree for $23, 29, 84, 15, 58, 19$:



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Remainder tree
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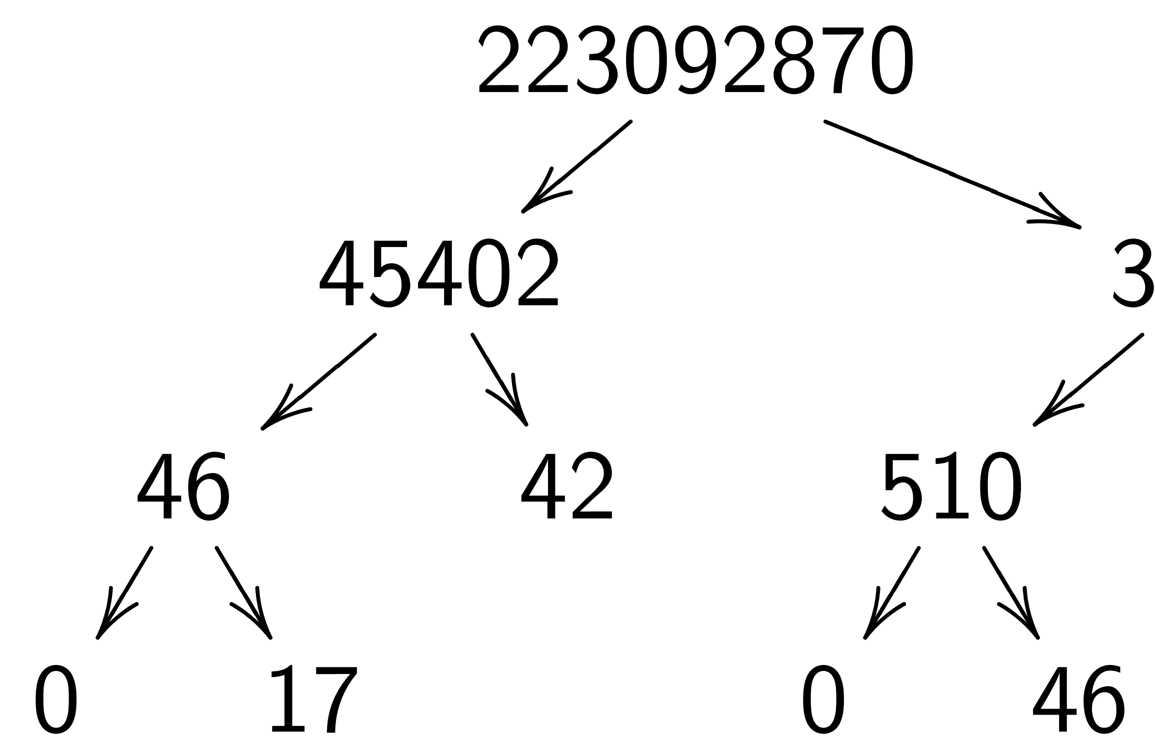
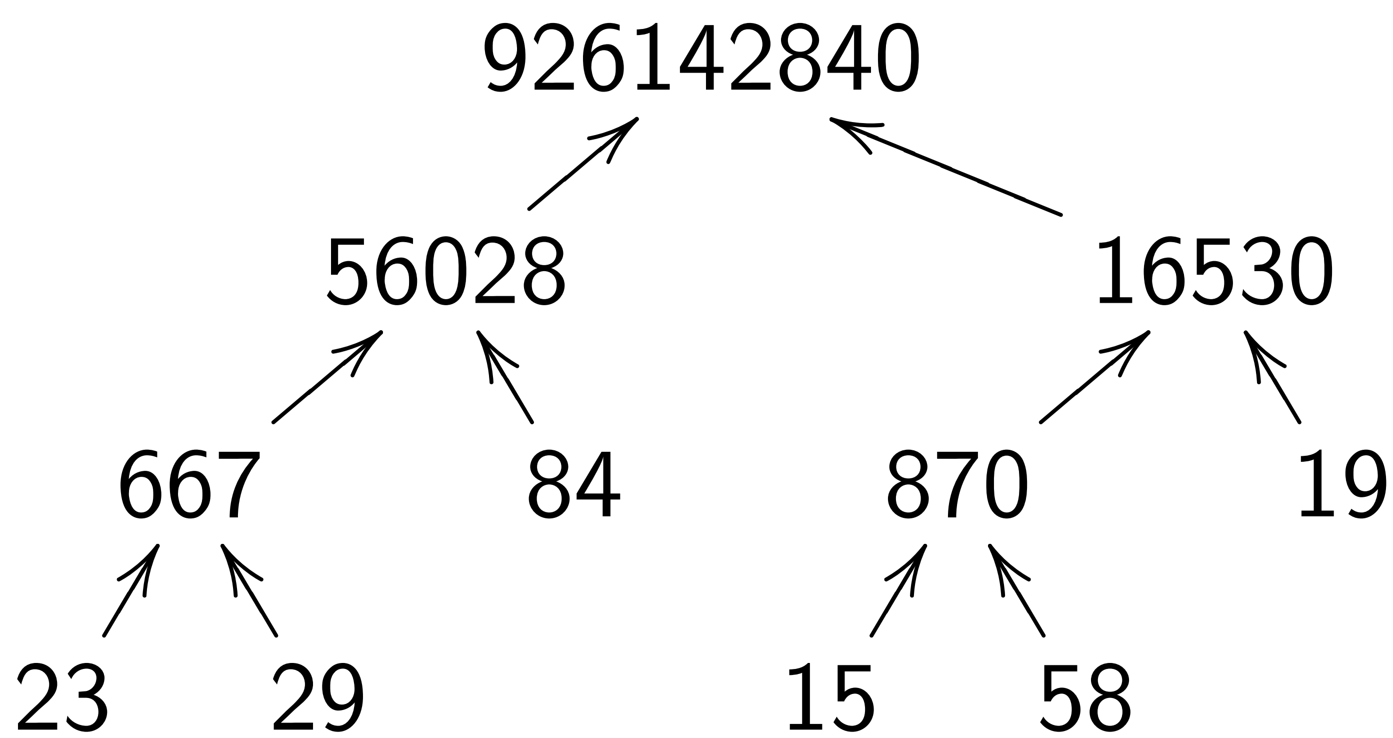
one node $r \bmod t$ for each n
in product tree of $x_1, x_2, \ldots$

e.g. remainder tree of
$223092870, 23, 29, 84, 15, 58$



e.g. tree for 23, 29, 84, 15, 58, 19:



Tree has $\leq (\lg b)^{1+o(1)}$ levels.
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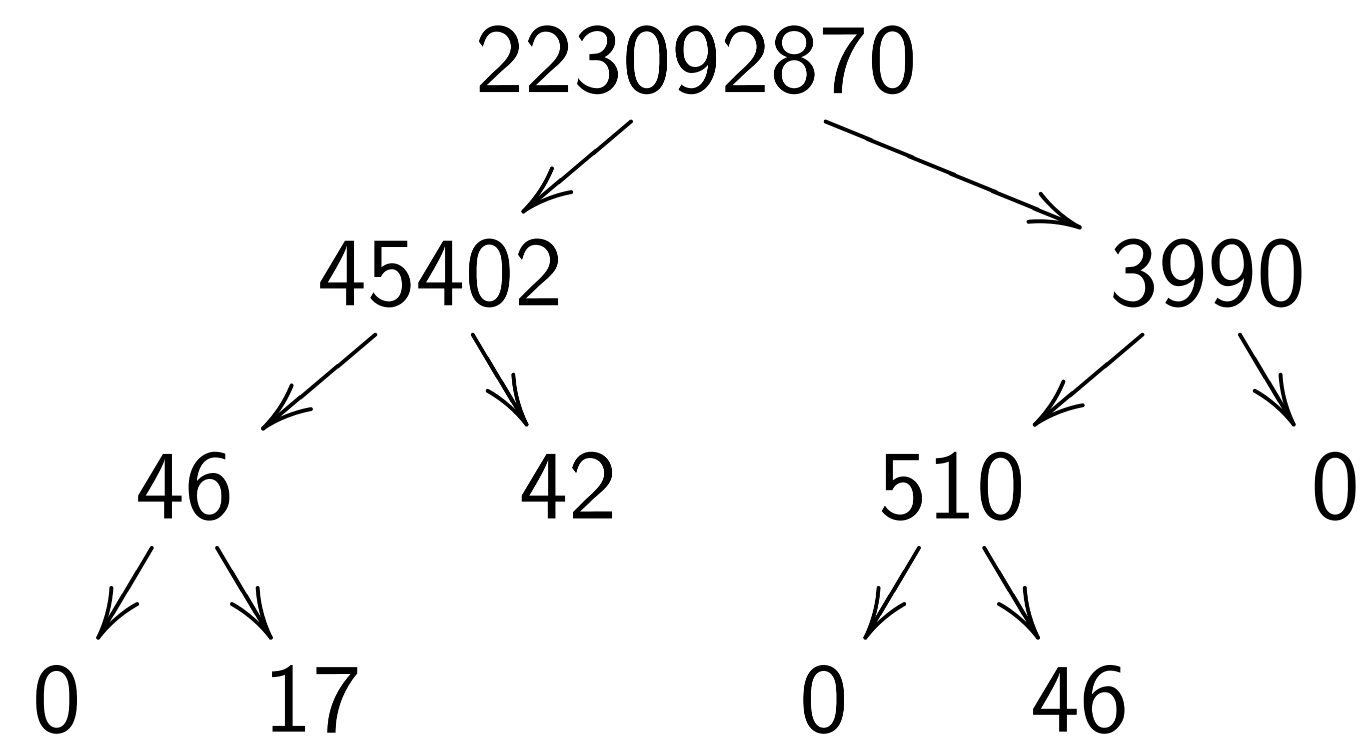
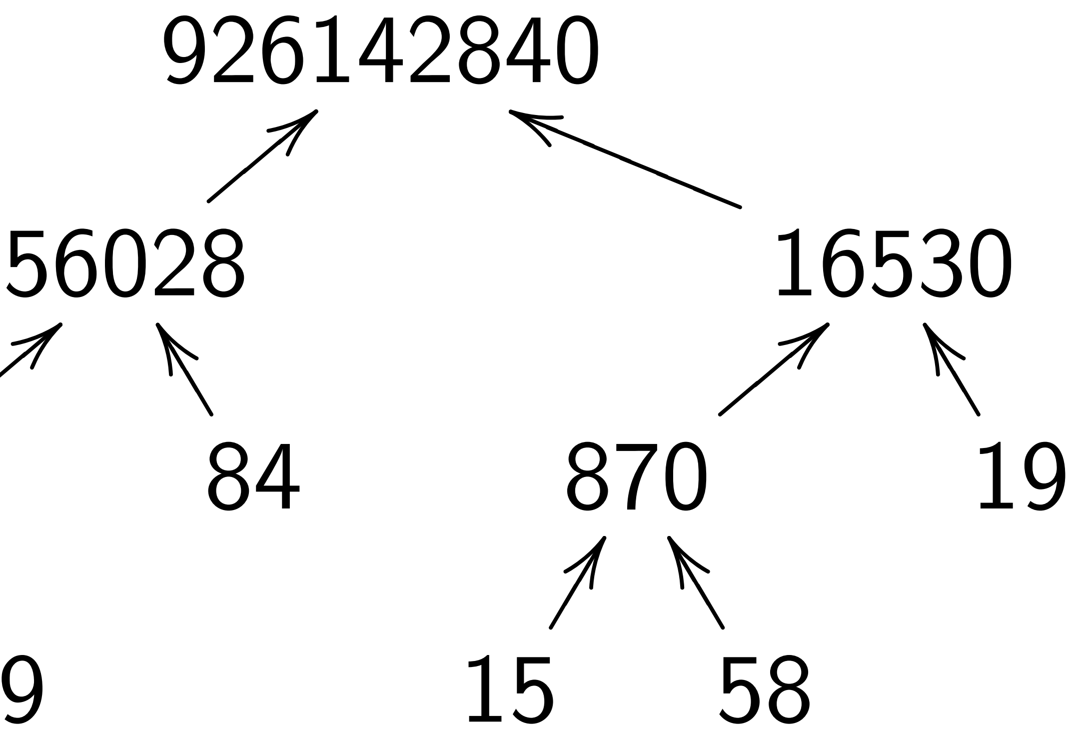


for 23, 29, 84, 15, 58, 19:



s $\leq (\lg b)^{1+o(1)}$ levels.
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Remainder trees

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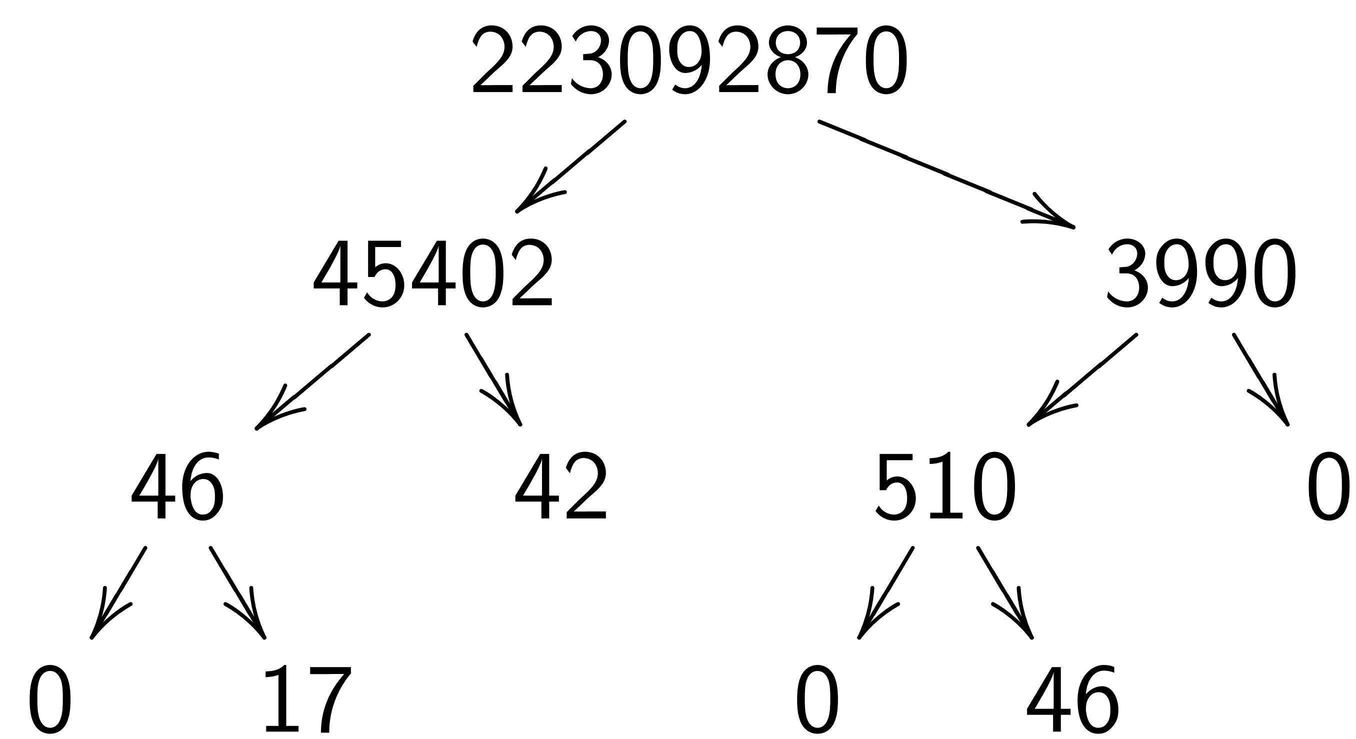

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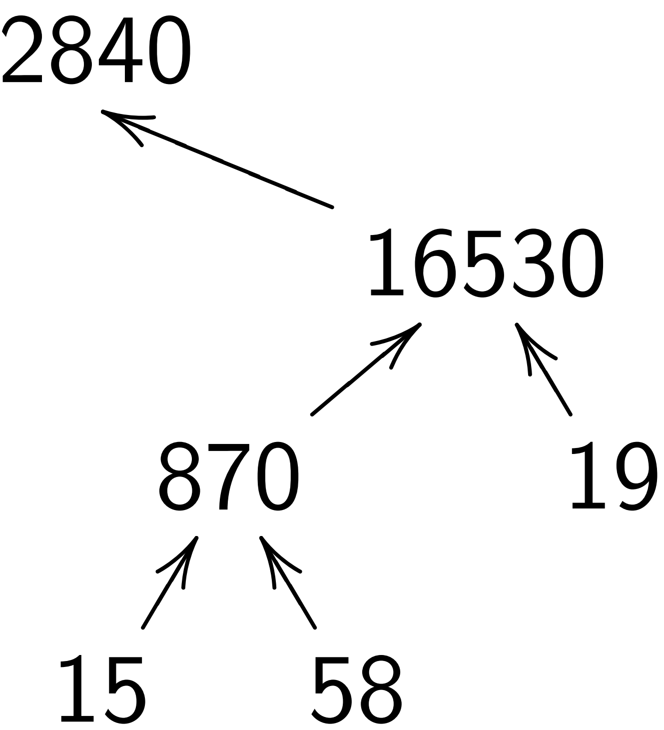

$+o(1)$ levels.

$b(\lg b)^{0+o(1)}$ bits.

$+o(1)$

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Remainder trees

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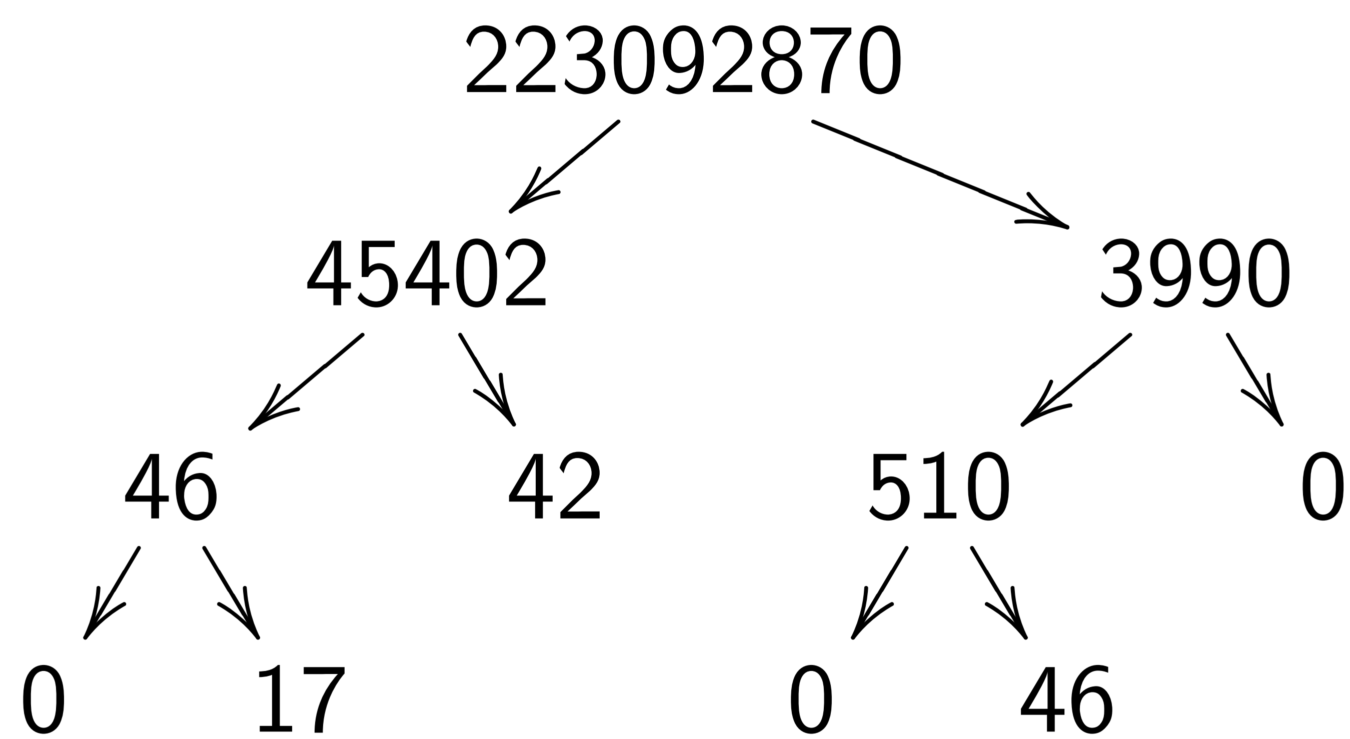

Time $\le b(\lg b)^{2+o}$

Given $r \in \mathbf{Z}$ and
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Remainder trees

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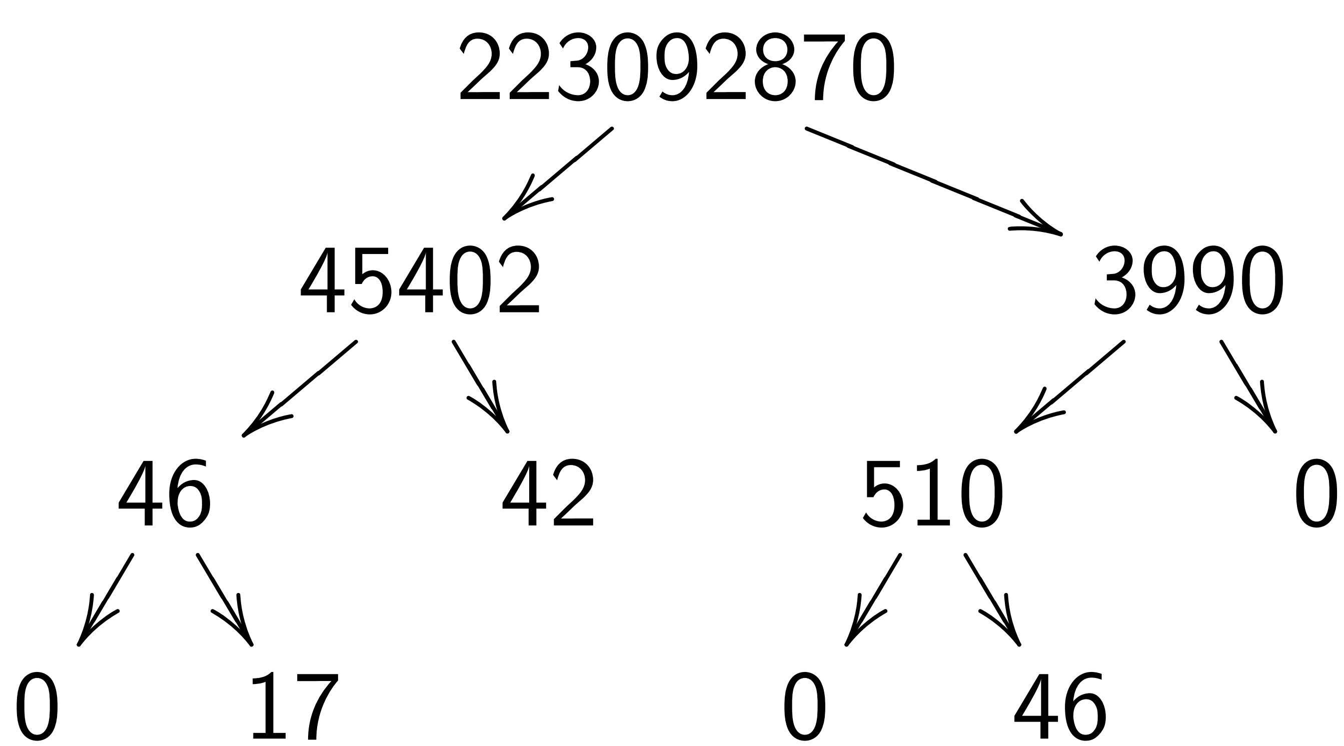

Time $\le b(\lg b)^{2+o(1)}$:
Given $r \in \mathbf{Z}$ and
nonzero $x_1, \ldots, x_n \in \mathbf{Z}$,
compute remainder tree
of $r, x_1, \ldots, x_n$.

In particular, compute
$r \bmod x_1, \ldots, r \bmod x_n$.

In particular, see which of
$x_1, \ldots, x_n$ divide r.

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for "single precision" x_i's,
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Remainder trees

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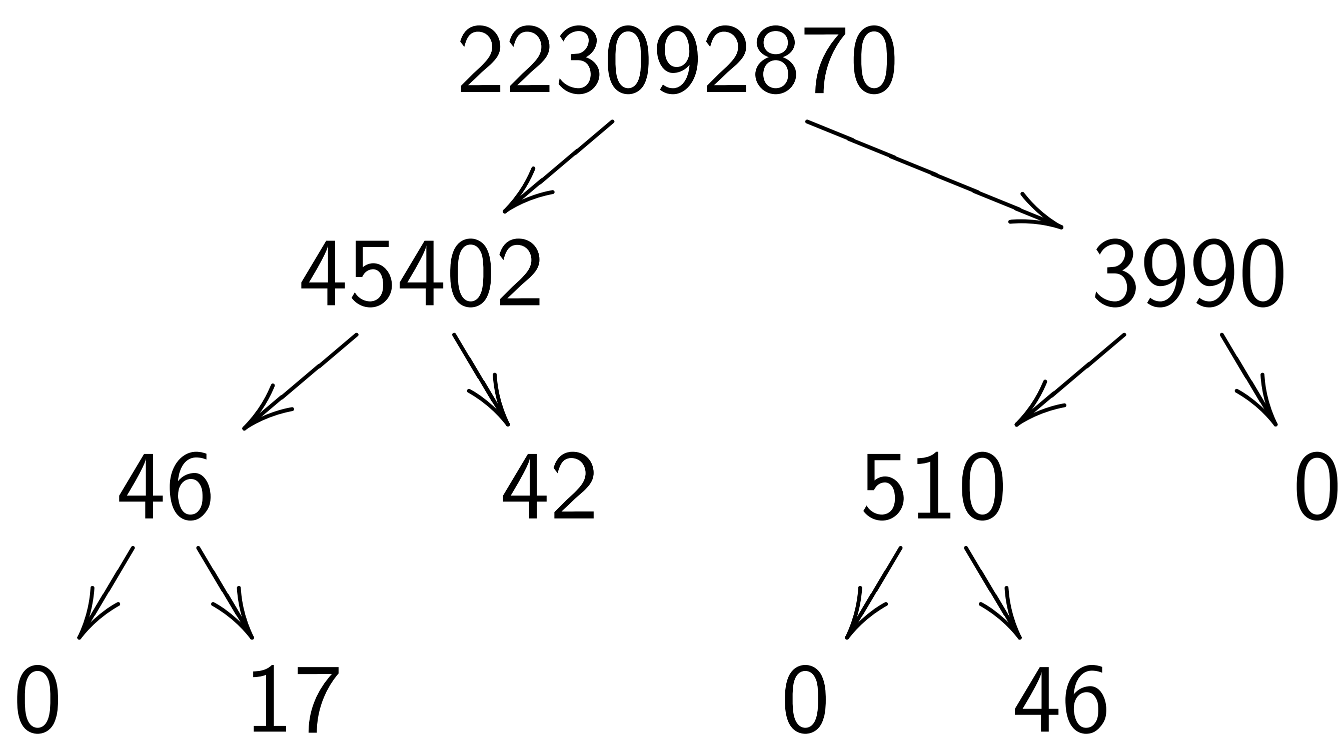

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(1972 Moenck Borodin,
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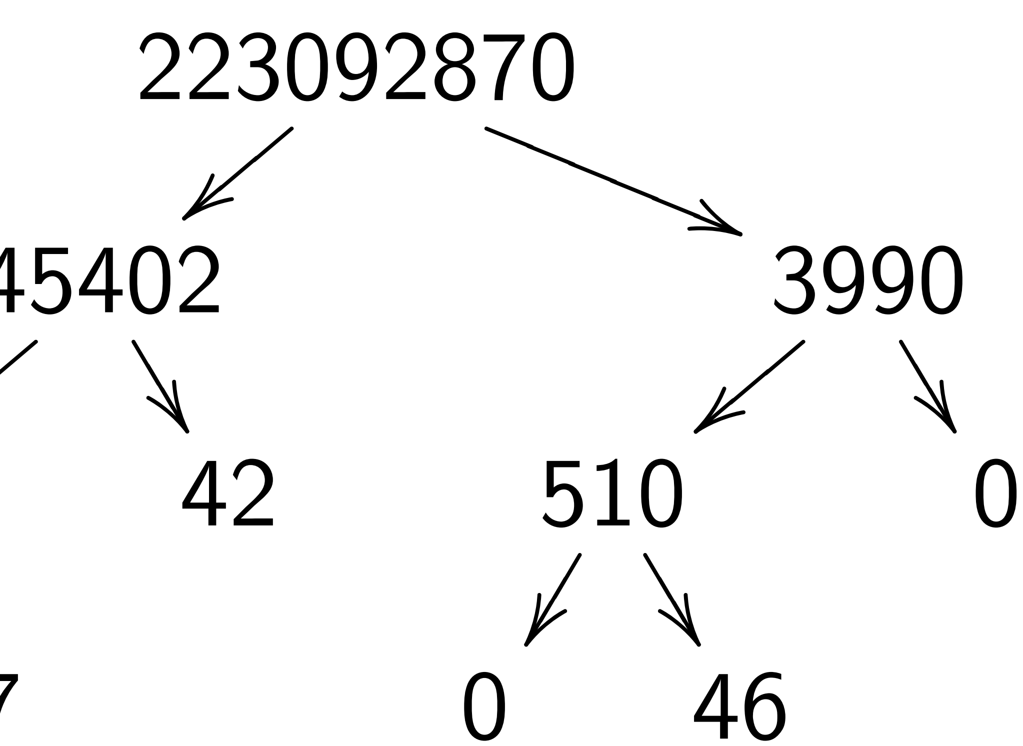

Time $\leq b(\lg b)^{2+o(1)}$:
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Time $\leq$
Given x_1
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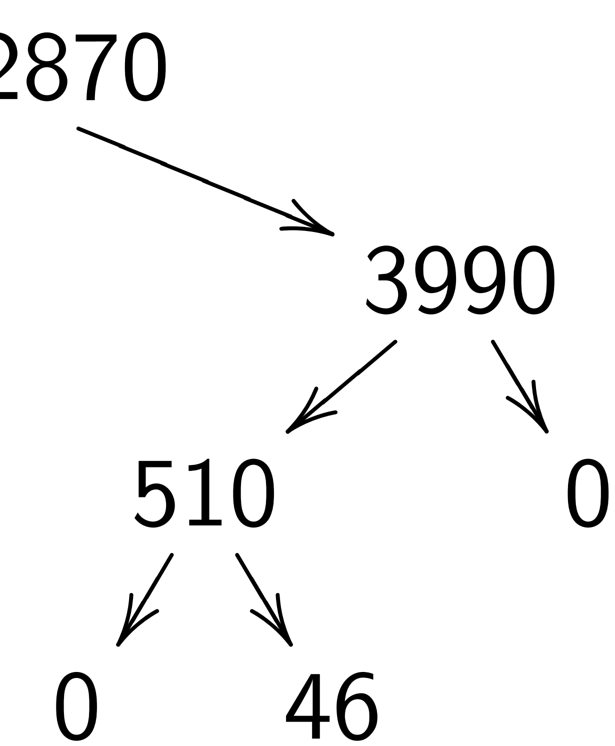

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(1972 Moenck Borodin,
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whatever exactly that means)

Small primes, unic

Time $\le b(\lg b)^{2+o}$
Given $x_1, x_2, \ldots,$
finite set $Q \subseteq \mathbf{Z} -$
$\{p \in Q : x_1 x_2 \cdots$

In particular, wher
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Algorithm:
1. Use a product t
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Time $\le b(\lg b)^{2+o(1)}$:
Given $r \in \mathbf{Z}$ and
nonzero $x_1, \ldots, x_n \in \mathbf{Z}$,
compute remainder tree
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(1972 Moenck Borodin,
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Small primes, union

Time $\le b(\lg b)^{2+o(1)}$:
Given $x_1, x_2, \ldots, x_n \in \mathbf{Z}$ an
finite set $Q \subseteq \mathbf{Z} - \{0\}$, com
$\{p \in Q : x_1 x_2 \cdots x_n \bmod p$

In particular, when p is prim
see whether p divides
any of $x_1, x_2, \ldots, x_n$.

Algorithm:

1. Use a product tree to
   compute $r = x_1 x_2 \cdots x_n$
2. Use a remainder tree to s
   which $p \in Q$ divide r.

Time $\le b(\lg b)^{2+o(1)}$:
Given $r \in \mathbf{Z}$ and nonzero $x_1, \ldots, x_n \in \mathbf{Z}$, compute remainder tree of $r, x_1, \ldots, x_n$.

In particular, compute $r \bmod x_1, \ldots, r \bmod x_n$.

In particular, see which of $x_1, \ldots, x_n$ divide r.

(1972 Moenck Borodin, for "single precision" x_i's, whatever exactly that means)

Small primes, union

Time $\le b(\lg b)^{2+o(1)}$:
Given $x_1, x_2, \ldots, x_n \in \mathbf{Z}$ and finite set $Q \subseteq \mathbf{Z} - \{0\}$, compute $\{p \in Q : x_1 x_2 \cdots x_n \bmod p = 0\}$.

In particular, when p is prime, see whether p divides any of $x_1, x_2, \ldots, x_n$.

Algorithm:

1. Use a product tree to compute $r = x_1 x_2 \cdots x_n$.
2. Use a remainder tree to see which $p \in Q$ divide r.

$b(\lg b)^{2+o(1)}$:
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Time $\le b(\lg b)^{2+o(1)}$:
Given $x_1, x_2, \ldots, x_n \in$ **Z** and
finite set $Q \subseteq$ **Z** $- \{0\}$, compute
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Small pr

Time $\le$
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Small primes, union

Time $\le b(\lg b)^{2+o(1)}$:
Given $x_1, x_2, \ldots, x_n \in$ **Z** and finite set $Q \subseteq$ **Z** $- \{0\}$, compute $\{p \in Q : x_1 x_2 \cdots x_n \bmod p = 0\}$.

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Small primes, sepa

Time $\le b(\lg b)^{3+o}$
Given $x_1, x_2, \ldots, x$
finite set Q of prin
compute $\{p \in Q :$
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(2000 Bernstein)

Algorithm for $n \ge$

1. Replace Q with
   $\{p \in Q : x_1 \cdots$
2. If $n = 1$, print
3. Recurse on $x_1,$
4. Recurse on $x_{\lceil n}$

s)

Small primes, union

Time $\leq b(\lg b)^{2+o(1)}$:
Given $x_1, x_2, \ldots, x_n \in \mathbf{Z}$ and finite set $Q \subseteq \mathbf{Z} - \{0\}$, compute $\{p \in Q : x_1 x_2 \cdots x_n \bmod p = 0\}$.

In particular, when p is prime, see whether p divides any of $x_1, x_2, \ldots, x_n$.

Algorithm:

1. Use a product tree to compute $r = x_1 x_2 \cdots x_n$.
2. Use a remainder tree to see which $p \in Q$ divide r.

Small primes, separately

Time $\leq b(\lg b)^{3+o(1)}$:
Given $x_1, x_2, \ldots, x_n \in \mathbf{Z}$ an
finite set Q of primes,
compute $\{p \in Q : x_1 \bmod p$
$\ldots, \{p \in Q : x_n \bmod p = 0$
(2000 Bernstein)

Algorithm for $n \geq 1$:

1. Replace Q with
   $\{p \in Q : x_1 \cdots x_n \bmod p$
2. If $n = 1$, print Q and sto
3. Recurse on $x_1, \ldots, x_{\lceil n/2}$
4. Recurse on $x_{\lceil n/2 \rceil + 1}, \ldots$

Small primes, union

Time $\le b(\lg b)^{2+o(1)}$:
Given $x_1, x_2, \ldots, x_n \in \mathbf{Z}$ and finite set $Q \subseteq \mathbf{Z} - \{0\}$, compute $\{p \in Q : x_1 x_2 \cdots x_n \bmod p = 0\}$.

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Algorithm:

1. Use a product tree to compute $r = x_1 x_2 \cdots x_n$.
2. Use a remainder tree to see which $p \in Q$ divide r.

Small primes, separately

Time $\le b(\lg b)^{3+o(1)}$:
Given $x_1, x_2, \ldots, x_n \in \mathbf{Z}$ and finite set Q of primes, compute $\{p \in Q : x_1 \bmod p = 0\}$, $\ldots$, $\{p \in Q : x_n \bmod p = 0\}$.
(2000 Bernstein)

Algorithm for $n \ge 1$:

1. Replace Q with $\{p \in Q : x_1 \cdots x_n \bmod p = 0\}$.
2. If $n = 1$, print Q and stop.
3. Recurse on $x_1, \ldots, x_{\lceil n/2 \rceil}, Q$.
4. Recurse on $x_{\lceil n/2 \rceil + 1}, \ldots, x_n, Q$.

rimes, union

$b(\lg b)^{2+o(1)}$:

$_1, x_2, \ldots, x_n \in \mathbf{Z}$ and

t $Q \subseteq \mathbf{Z} - \{0\}$, compute

: $x_1 x_2 \cdots x_n \bmod p = 0\}$.

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Small primes, separately

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Given $x_1, x_2, \ldots, x_n \in \mathbf{Z}$ and

finite set Q of primes,

compute $\{p \in Q : x_1 \bmod p = 0\}$,

$\ldots$, $\{p \in Q : x_n \bmod p = 0\}$.

(2000 Bernstein)

Algorithm for $n \ge 1$:

1. Replace Q with $\{p \in Q : x_1 \cdots x_n \bmod p = 0\}$.
2. If $n = 1$, print Q and stop.
3. Recurse on $x_1, \ldots, x_{\lceil n/2 \rceil}, Q$.
4. Recurse on $x_{\lceil n/2 \rceil + 1}, \ldots, x_n, Q$.

Factor

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Small primes, separately

Time $\le b(\lg b)^{3+o(1)}$:
Given $x_1, x_2, \ldots, x_n \in \mathbf{Z}$ and
finite set Q of primes,
compute $\{p \in Q : x_1 \bmod p = 0\}$,
$\ldots$, $\{p \in Q : x_n \bmod p = 0\}$.
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4. Recurse on $x_{\lceil n/2 \rceil + 1}, \ldots, x_n, Q$.

Factor 2543, 676
over {2, 3, 5, 7,

2543, 6766
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Small primes, separately

Time $\le b(\lg b)^{3+o(1)}$:
Given $x_1, x_2, \ldots, x_n \in \mathbf{Z}$ and finite set Q of primes, compute $\{p \in Q : x_1 \bmod p = 0\}$, $\ldots$, $\{p \in Q : x_n \bmod p = 0\}$.
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2. If $n = 1$, print Q and stop.
3. Recurse on $x_{1}, \ldots, x_{\lceil n/2 \rceil}, Q$.
4. Recurse on $x_{\lceil n/2 \rceil + 1}, \ldots, x_n, Q$.

Factor $2543, 6766, 8967, 75$
over $\{\underline{2}, \underline{3}, 5, \underline{7}, 11, 13, \underline{17}$

$2543, 6766$ over $\underline{2}, 3, 7, \underline{17}$

$8967, 759$ over $\underline{2}, \underline{3}, \underline{7}, 1$

2543 over $2, 17$

6766 over $\underline{2}, \underline{17}$

8967 over $2, \underline{3}, \underline{7}$

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Each level has $\le b(\lg b)^{0+o(1}$

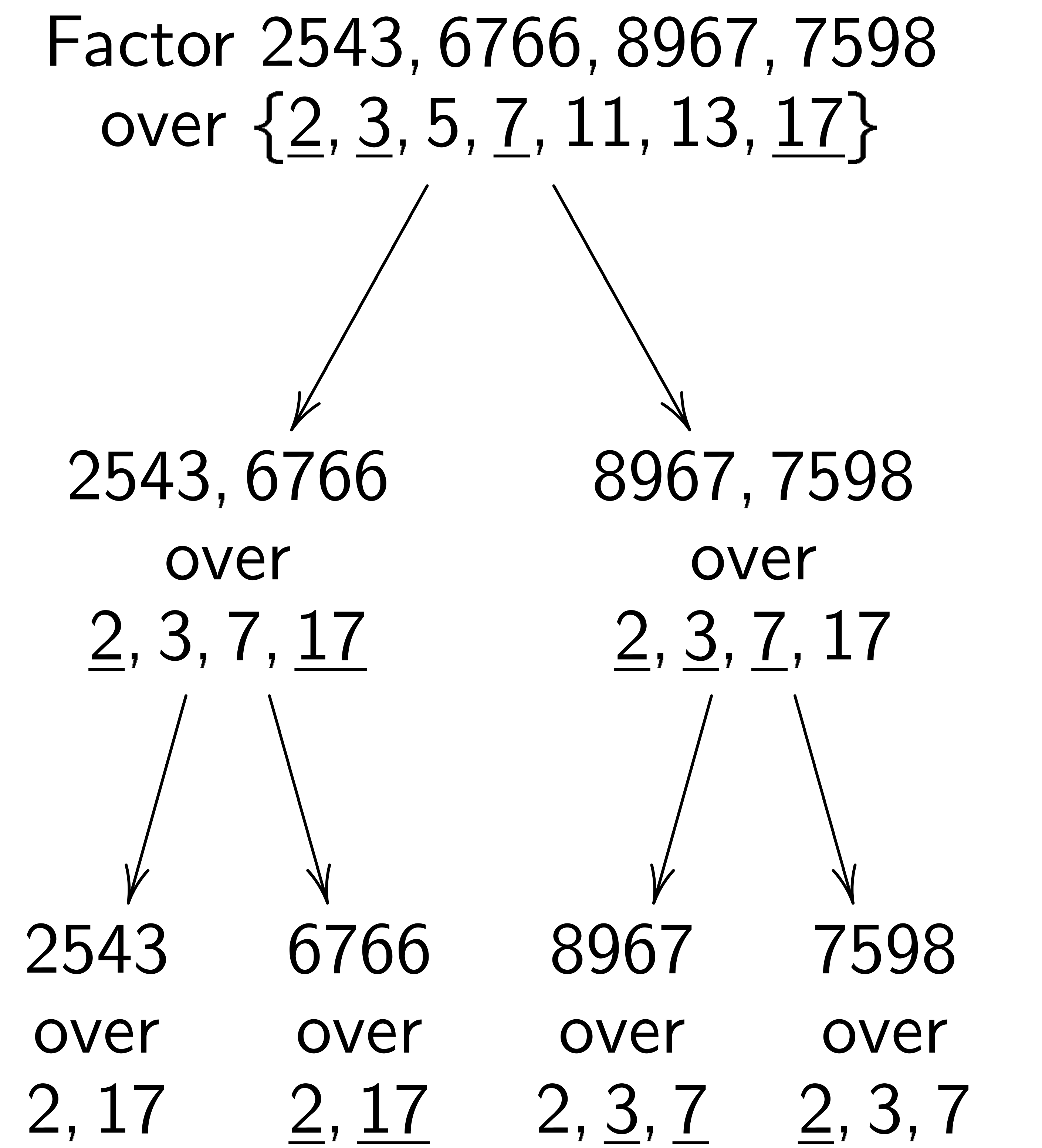
Small primes, separately

Time $\le b(\lg b)^{3+o(1)}$:

Given $x_1, x_2, \ldots, x_n \in \mathbf{Z}$ and finite set Q of primes, compute $\{p \in Q : x_1 \bmod p = 0\}$, $\ldots$, $\{p \in Q : x_n \bmod p = 0\}$.
(2000 Bernstein)

Algorithm for $n \ge 1$:

1. Replace Q with $\{p \in Q : x_1 \cdots x_n \bmod p = 0\}$.
2. If $n = 1$, print Q and stop.
3. Recurse on $x_1, \ldots, x_{\lceil n/2 \rceil}, Q$.
4. Recurse on $x_{\lceil n/2 \rceil + 1}, \ldots, x_n, Q$.



Each level has $\le b(\lg b)^{0+o(1)}$ bits.

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$b(\lg b)^{3+o(1)}$:

$_1, x_2, \ldots, x_n \in \mathbf{Z}$ and

t Q of primes,

e $\{p \in Q : x_1 \bmod p = 0\}$,

$\in Q : x_n \bmod p = 0\}$.

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m for $n \geq 1$:

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$Q : x_1 \cdots x_n \bmod p = 0\}$.

$= 1$, print Q and stop.

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rse on $x_{\lceil n/2 \rceil + 1}, \ldots, x_n, Q$.

Factor $2543, 6766, 8967, 7598$
over $\{\underline{2}, \underline{3}, 5, \underline{7}, 11, 13, \underline{17}\}$

$2543, 6766$ over $\underline{2}, 3, 7, \underline{17}$

$8967, 7598$ over $\underline{2}, \underline{3}, \underline{7}, 17$

2543 over $2, 17$

6766 over $\underline{2}, \underline{17}$

8967 over $2, \underline{3}, \underline{7}$

7598 over $\underline{2}, 3, 7$

Each level has $\leq b(\lg b)^{0+o(1)}$ bits.

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$x_n \bmod p = 0\}$.

Q and stop.

$\ldots, x_{\lceil n/2 \rceil}, Q$.

$/2\rceil+1}, \ldots, x_n, Q$.

Factor $2543, 6766, 8967, 7598$
over $\{\underline{2}, \underline{3}, 5, \underline{7}, 11, 13, \underline{17}\}$

$2543, 6766$ over $\underline{2}, 3, 7, \underline{17}$

$8967, 7598$ over $\underline{2}, \underline{3}, \underline{7}, 17$

2543 over $2, 17$

6766 over $\underline{2}, \underline{17}$

8967 over $2, \underline{3}, \underline{7}$

7598 over $\underline{2}, 3, 7$

Each level has $\le b(\lg b)^{0+o(1)}$ bits.

Exponents of a sm

Time $\le b(\lg b)^{2+o}$

Given nonzero p, x

find $e, p^e, x/p^e$ wi

Algorithm:

1. If $x \bmod p \ne 0$
   Print $0, 1, x$ an
2. Find $f, (p^2)^f, r$
   with maximal f
3. If $r \bmod p = 0$:
   $2f + 2, (p^2)^f p^2$
4. Print $2f + 1, ($

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Factor $2543, 6766, 8967, 7598$
over $\{\underline{2}, \underline{3}, 5, \underline{7}, 11, 13, \underline{17}\}$

$2543, 6766$ over $\underline{2}, 3, 7, \underline{17}$

$8967, 7598$ over $\underline{2}, \underline{3}, \underline{7}, 17$

2543 over $2, 17$

6766 over $\underline{2}, \underline{17}$

8967 over $2, \underline{3}, \underline{7}$

7598 over $\underline{2}, 3, 7$

Each level has $\le b(\lg b)^{0+o(1)}$ bits.

Exponents of a small prime

Time $\le b(\lg b)^{2+o(1)}$:
Given nonzero $p, x \in \mathbf{Z}$,
find $e, p^e, x/p^e$ with maxima

Algorithm:

1. If $x \bmod p \ne 0$:
   Print $0, 1, x$ and stop.
2. Find $f, (p^2)^f, r = (x/p)/$
   with maximal f.
3. If $r \bmod p = 0$: Print
   $2f + 2, (p^2)^f p^2, r/p$ and
4. Print $2f + 1, (p^2)^f p, r$.

Factor 2543, 6766, 8967, 7598
over $\{\underline{2}, \underline{3}, 5, \underline{7}, 11, 13, \underline{17}\}$

- 2543, 6766 over $\underline{2}, 3, 7, \underline{17}$
  - 2543 over 2, 17
  - 6766 over $\underline{2}, \underline{17}$
- 8967, 7598 over $\underline{2}, \underline{3}, \underline{7}, 17$
  - 8967 over $2, \underline{3}, \underline{7}$
  - 7598 over $\underline{2}, 3, 7$

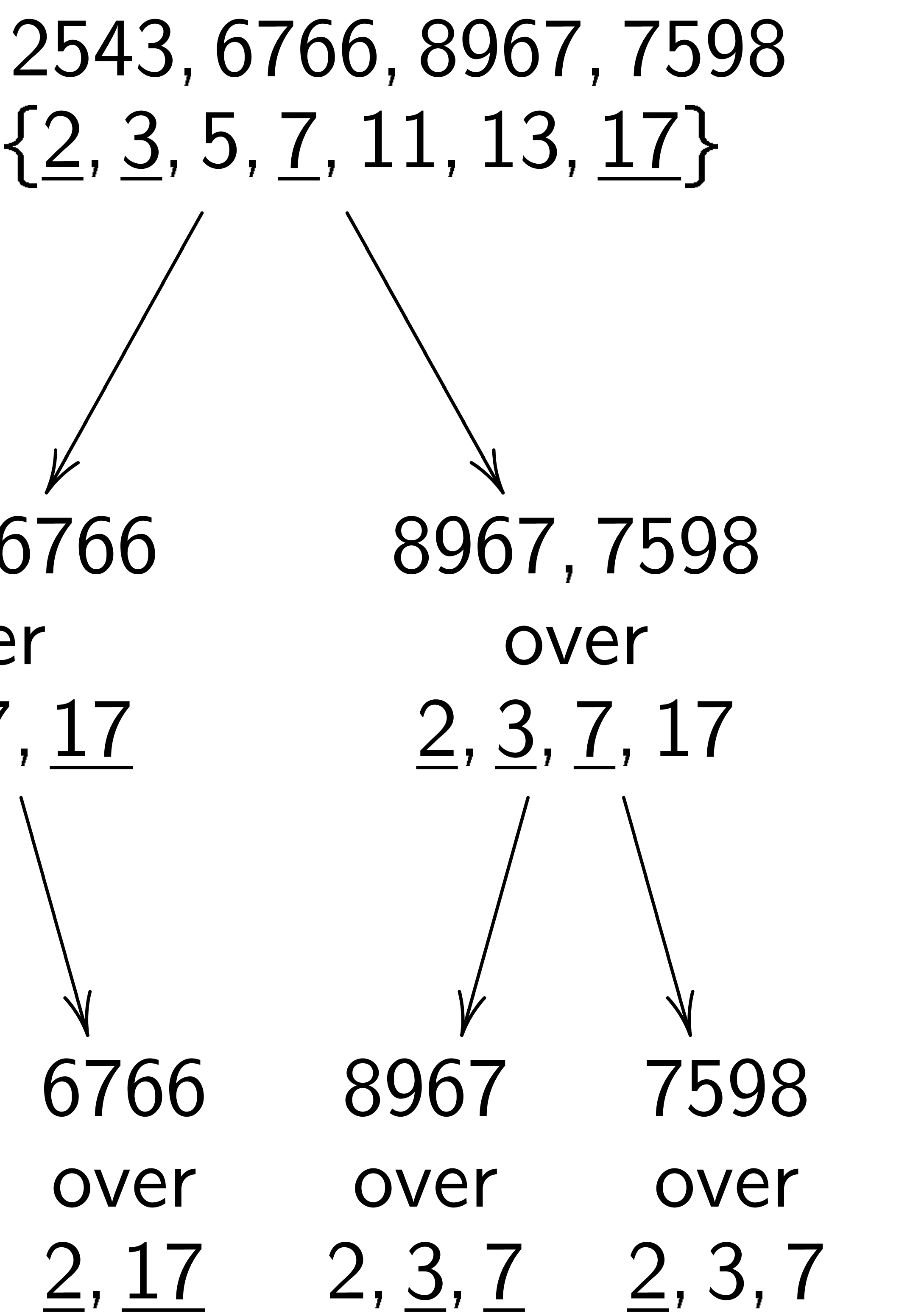
Each level has $\leq b(\lg b)^{0+o(1)}$ bits.

Exponents of a small prime

Time $\leq b(\lg b)^{2+o(1)}$:
Given nonzero $p, x \in \mathbf{Z}$,
find $e, p^e, x/p^e$ with maximal e.

Algorithm:

1. If $x \bmod p \neq 0$:
   Print $0, 1, x$ and stop.
2. Find $f, (p^2)^f, r = (x/p)/(p^2)^f$
   with maximal f.
3. If $r \bmod p = 0$: Print
   $2f + 2, (p^2)^f p^2, r/p$ and stop.
4. Print $2f + 1, (p^2)^f p, r$.



vel has $\leq b(\lg b)^{0+o(1)}$ bits.

Exponents of a small prime

Time $\leq b(\lg b)^{2+o(1)}$:
Given nonzero $p, x \in \mathbf{Z}$,
find $e, p^e, x/p^e$ with maximal e.

Algorithm:

1. If $x \bmod p \neq 0$:
   Print $0, 1, x$ and stop.
2. Find $f, (p^2)^f, r = (x/p)/(p^2)^f$
   with maximal f.
3. If $r \bmod p = 0$: Print
   $2f + 2, (p^2)^f p^2, r/p$ and stop.
4. Print $2f + 1, (p^2)^f p, r$.

Exponen

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$e, \prod_{p \in Q}$

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   $\{p \in$
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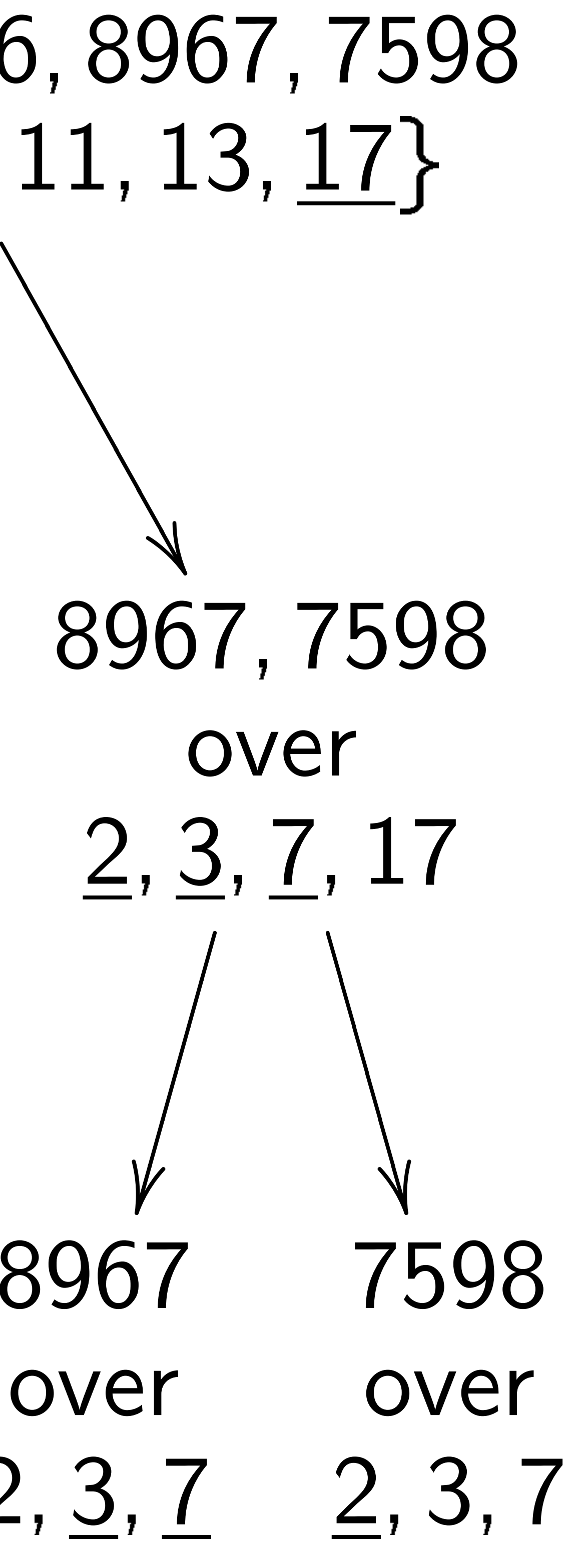

$b(\lg b)^{0+o(1)}$ bits.

Exponents of a small prime

Time $\le b(\lg b)^{2+o(1)}$:

Given nonzero $p, x \in \mathbf{Z}$,

find $e, p^e, x/p^e$ with maximal e.

Algorithm:

1. If $x \bmod p \ne 0$:
   Print $0, 1, x$ and stop.
2. Find $f, (p^2)^f, r = (x/p)/(p^2)^f$ with maximal f.
3. If $r \bmod p = 0$: Print $2f + 2, (p^2)^f p^2, r/p$ and stop.
4. Print $2f + 1, (p^2)^f p, r$.

Exponents of smal

Time $\le b(\lg b)^{3+o}$

Given finite set Q

and nonzero $x \in \mathbf{Z}$

$e, \prod_{p \in Q} p^{e(p)}, x/\prod$

Algorithm:

1. Replace Q with
   $\{p \in Q : x$ moc
2. Find maximal f
   $s = \prod(p^2)^{f(p^2)}$
3. Find $T = \{p \in$
4. Output $e, s\prod_p$
   where $e(p) = 2$

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1) bits.

Exponents of a small prime

Time $\le b(\lg b)^{2+o(1)}$:
Given nonzero $p, x \in \mathbf{Z}$,
find $e, p^e, x/p^e$ with maximal e.

Algorithm:

1. If $x \bmod p \ne 0$:
   Print $0, 1, x$ and stop.
2. Find $f, (p^2)^f, r = (x/p)/(p^2)^f$
   with maximal f.
3. If $r \bmod p = 0$: Print
   $2f+2, (p^2)^f p^2, r/p$ and stop.
4. Print $2f+1, (p^2)^f p, r$.

Exponents of small primes

Time $\le b(\lg b)^{3+o(1)}$:
Given finite set Q of primes
and nonzero $x \in \mathbf{Z}$, find ma
$e, \prod_{p\in Q} p^{e(p)}, x/\prod_{p\in Q} p^{e(p}$

Algorithm:

1. Replace Q with
   $\{p \in Q : x \bmod p = 0\}$.
2. Find maximal f, s, r with
   $s = \prod (p^2)^{f(p^2)}$, $r = (x/$
3. Find $T = \{p \in Q : r \bmod$
4. Output $e, s \prod_{p \in T} p, r/$
   where $e(p) = 2f(p^2) + [$

Exponents of a small prime

Time $\le b(\lg b)^{2+o(1)}$:
Given nonzero $p, x \in \mathbf{Z}$,
find $e, p^e, x/p^e$ with maximal e.

Algorithm:

1. If $x \bmod p \ne 0$:
   Print $0, 1, x$ and stop.
2. Find $f, (p^2)^f, r = (x/p)/(p^2)^f$
   with maximal f.
3. If $r \bmod p = 0$: Print
   $2f + 2, (p^2)^f p^2, r/p$ and stop.
4. Print $2f + 1, (p^2)^f p, r$.

Exponents of small primes

Time $\le b(\lg b)^{3+o(1)}$:
Given finite set Q of primes
and nonzero $x \in \mathbf{Z}$, find maximal
$e, \prod_{p \in Q} p^{e(p)}, x/\prod_{p \in Q} p^{e(p)}$.

Algorithm:

1. Replace Q with
   $\{p \in Q : x \bmod p = 0\}$.
2. Find maximal f, s, r with
   $s = \prod (p^2)^{f(p^2)}$, $r = (x/\prod p)/s$.
3. Find $T = \{p \in Q : r \bmod p = 0\}$.
4. Output $e, s\prod_{p \in T} p, r/\prod_{p \in T} p$
   where $e(p) = 2f(p^2) + [p \in T]$.

nts of a small prime

$b(\lg b)^{2+o(1)}$:

onzero $p, x \in \mathbf{Z}$,

$^e, x/p^e$ with maximal e.

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$0, 1, x$ and stop.

$f, (p^2)^f, r = (x/p)/(p^2)^f$

maximal f.

nod $p = 0$: Print

$2, (p^2)^f p^2, r/p$ and stop.

$2f + 1, (p^2)^f p, r$.

Exponents of small primes

Time $\leq b(\lg b)^{3+o(1)}$:
Given finite set Q of primes
and nonzero $x \in \mathbf{Z}$, find maximal
$e, \prod_{p \in Q} p^{e(p)}, x / \prod_{p \in Q} p^{e(p)}$.

Algorithm:

1. Replace Q with $\{p \in Q : x \bmod p = 0\}$.
2. Find maximal f, s, r with $s = \prod (p^2)^{f(p^2)}$, $r = (x / \prod p)/s$.
3. Find $T = \{p \in Q : r \bmod p = 0\}$.
4. Output $e, s \prod_{p \in T} p, r / \prod_{p \in T} p$ where $e(p) = 2f(p^2) + [p \in T]$.

Smooth

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$= (x/p)/(p^2)^f$

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: Print

$^2, r/p$ and stop.

$p^2)^f p, r$.

Exponents of small primes

Time $\le b(\lg b)^{3+o(1)}$:
Given finite set Q of primes
and nonzero $x \in \mathbf{Z}$, find maximal
$e, \prod_{p \in Q} p^{e(p)}, x / \prod_{p \in Q} p^{e(p)}$.

Algorithm:

1. Replace Q with
   $\{p \in Q : x \bmod p = 0\}$.
2. Find maximal f, s, r with
   $s = \prod (p^2)^{f(p^2)}$, $r = (x / \prod p)/s$.
3. Find $T = \{p \in Q : r \bmod p = 0\}$.
4. Output $e, s \prod_{p \in T} p, r / \prod_{p \in T} p$
   where $e(p) = 2f(p^2) + [p \in T]$.

Smooth parts, old

Time $\le b(\lg b)^{3+o}$
Given nonzero x_1,
and finite set Q of
compute Q-smoot
Q-smooth part of
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In particular, see w
$x_1, x_2, \ldots, x_n$ are

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$(p^2)^f$

stop.

Exponents of small primes

Time $\le b(\lg b)^{3+o(1)}$:
Given finite set Q of primes and nonzero $x \in \mathbf{Z}$, find maximal $e, \prod_{p \in Q} p^{e(p)}, x / \prod_{p \in Q} p^{e(p)}$.

Algorithm:

1. Replace Q with $\{p \in Q : x \bmod p = 0\}$.
2. Find maximal f, s, r with $s = \prod (p^2)^{f(p^2)}$, $r = (x / \prod p)/s$.
3. Find $T = \{p \in Q : r \bmod p = 0\}$.
4. Output $e, s \prod_{p \in T} p, r / \prod_{p \in T} p$ where $e(p) = 2f(p^2) + [p \in T]$.

Smooth parts, old approach

Time $\le b(\lg b)^{3+o(1)}$:
Given nonzero $x_1, x_2, \ldots, x_{n}$
and finite set Q of primes,
compute Q-smooth part of x
Q-smooth part of x_2, ...,
Q-smooth part of x_n.

Q-smooth means product of powers of elements of Q.

Q-smooth part means largest Q-smooth divisor.
In particular, see which of $x_1, x_2, \ldots, x_n$ are smooth.

Exponents of small primes

Time $\le b(\lg b)^{3+o(1)}$:
Given finite set Q of primes and nonzero $x \in \mathbf{Z}$, find maximal $e, \prod_{p\in Q} p^{e(p)}, x/\prod_{p\in Q} p^{e(p)}$.

Algorithm:

1. Replace Q with $\{p \in Q : x \bmod p = 0\}$.
2. Find maximal f, s, r with $s = \prod (p^2)^{f(p^2)}$, $r = (x/\prod p)/s$.
3. Find $T = \{p \in Q : r \bmod p = 0\}$.
4. Output $e, s\prod_{p\in T} p, r/\prod_{p\in T} p$ where $e(p) = 2f(p^2) + [p \in T]$.

Smooth parts, old approach

Time $\le b(\lg b)^{3+o(1)}$:
Given nonzero $x_1, x_2, \ldots, x_n \in \mathbf{Z}$ and finite set Q of primes, compute Q-smooth part of x_1, Q-smooth part of x_2, ..., Q-smooth part of x_n.

Q-smooth means product of powers of elements of Q.

Q-smooth part means largest Q-smooth divisor.
In particular, see which of $x_1, x_2, \ldots, x_n$ are smooth.

nts of small primes

$b(\lg b)^{3+o(1)}$:

nite set Q of primes

zero $x \in \mathbf{Z}$, find maximal

$p^{e(p)}, x/\prod_{p \in Q} p^{e(p)}$.

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$Q : x \bmod p = 0\}$.

maximal f, s, r with

$\prod(p^2)^{f(p^2)}$, $r = (x/\prod p)/s$.

$T = \{p \in Q : r \bmod p = 0\}$.

ut $e, s\prod_{p \in T} p, r/\prod_{p \in T} p$

e $e(p) = 2f(p^2) + [p \in T]$.

Smooth parts, old approach

Time $\le b(\lg b)^{3+o(1)}$:
Given nonzero $x_1, x_2, \ldots, x_n \in \mathbf{Z}$
and finite set Q of primes,
compute Q-smooth part of x_1,
Q-smooth part of x_2, ...,
Q-smooth part of x_n.

Q-smooth means product
of powers of elements of Q.

Q-smooth part means
largest Q-smooth divisor.
In particular, see which of
$x_1, x_2, \ldots, x_n$ are smooth.

Algorith

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   ..., $($
2. For e
   Find
   $s = \prod$
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$\prod_{p \in Q} p^{e(p)}$.

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d $p = 0\}$.

f, s, r with

, $r = (x / \prod p)/s$.

Q : r mod $p = 0\}$.

$_{\in T}\, p, r / \prod_{p \in T} p$

$2f(p^2) + [p \in T]$.

Smooth parts, old approach

Time $\leq b(\lg b)^{3+o(1)}$:
Given nonzero $x_1, x_2, \ldots, x_n \in \mathbf{Z}$
and finite set Q of primes,
compute Q-smooth part of x_1,
Q-smooth part of x_2, ...,
Q-smooth part of x_n.

Q-smooth means product
of powers of elements of Q.

Q-smooth part means
largest Q-smooth divisor.
In particular, see which of
$x_1, x_2, \ldots, x_n$ are smooth.

Algorithm:

1. Find $Q_1 = \{p :$
   ..., $Q_n = \{p :$
2. For each i sepa
   Find maximal e
   $s = \prod_{p \in Q_i} p^{e(p}$
   Print s.

e.g. factor 2543, 6
over $\{2, 3, 5, 7, 11,$
2543 over $\{\}$, smo
6766 over $\{2, 17\}$,
8967 over $\{3, 7\}$, s
7598 over $\{2\}$, sm

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$p = 0\}$.

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$p \in T]$.

Smooth parts, old approach

Time $\le b(\lg b)^{3+o(1)}$:
Given nonzero $x_1, x_2, \ldots, x_n \in \mathbf{Z}$
and finite set Q of primes,
compute Q-smooth part of x_1,
Q-smooth part of x_2, ...,
Q-smooth part of x_n.

Q-smooth means product
of powers of elements of Q.

Q-smooth part means
largest Q-smooth divisor.
In particular, see which of
$x_1, x_2, \ldots, x_n$ are smooth.

Algorithm:

1. Find $Q_1 = \{p : x_1 \bmod p$
   $\ldots,\ Q_n = \{p : x_n \bmod p$
2. For each i separately:
   Find maximal e, s, r with
   $s = \prod_{p \in Q_i} p^{e(p)}$, $r = x_{i/}$
   Print s.

e.g. factor 2543, 6766, 8967
over $\{2, 3, 5, 7, 11, 13, 17\}$:
2543 over $\{\}$, smooth part 1
6766 over $\{2, 17\}$, smooth p
8967 over $\{3, 7\}$, smooth pa
7598 over $\{2\}$, smooth part

Smooth parts, old approach

Time $\le b(\lg b)^{3+o(1)}$:

Given nonzero $x_1, x_2, \ldots, x_n \in \mathbf{Z}$ and finite set Q of primes, compute Q-smooth part of x_1, Q-smooth part of x_2, ..., Q-smooth part of x_n.

Q-smooth means product of powers of elements of Q.

Q-smooth part means largest Q-smooth divisor. In particular, see which of $x_1, x_2, \ldots, x_n$ are smooth.

Algorithm:

1. Find $Q_1 = \{p : x_1 \bmod p = 0\}$, $\ldots$, $Q_n = \{p : x_n \bmod p = 0\}$.
2. For each i separately: Find maximal e, s, r with $s = \prod_{p \in Q_i} p^{e(p)}$, $r = x_i/s$. Print s.

e.g. factor $2543, 6766, 8967, 7598$ over $\{2, 3, 5, 7, 11, 13, 17\}$:
2543 over $\{\}$, smooth part 1;
6766 over $\{2, 17\}$, smooth part 34;
8967 over $\{3, 7\}$, smooth part 147;
7598 over $\{2\}$, smooth part 2.

parts, old approach

$b(\lg b)^{3+o(1)}$:

onzero $x_1, x_2, \ldots, x_n \in \mathbf{Z}$

e set Q of primes,

e Q-smooth part of x_1,

th part of x_2, ...,

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ular, see which of

$\ldots, x_n$ are smooth.

Algorithm:

1. Find $Q_1 = \{p : x_1 \bmod p = 0\}$, ..., $Q_n = \{p : x_n \bmod p = 0\}$.
2. For each i separately:
   Find maximal e, s, r with $s = \prod_{p \in Q_i} p^{e(p)}$, $r = x_i/s$.
   Print s.

e.g. factor 2543, 6766, 8967, 7598 over $\{2, 3, 5, 7, 11, 13, 17\}$:

2543 over $\{\}$, smooth part 1;

6766 over $\{2, 17\}$, smooth part 34;

8967 over $\{3, 7\}$, smooth part 147;

7598 over $\{2\}$, smooth part 2.

Smooth

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x_2, ...,

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Algorithm:

1. Find $Q_1 = \{p : x_1 \bmod p = 0\}$, ..., $Q_n = \{p : x_n \bmod p = 0\}$.
2. For each i separately:
   Find maximal e, s, r with
   $s = \prod_{p \in Q_i} p^{e(p)}$, $r = x_i/s$.
   Print s.

e.g. factor 2543, 6766, 8967, 7598 over $\{2, 3, 5, 7, 11, 13, 17\}$:
2543 over $\{\}$, smooth part 1;
6766 over $\{2, 17\}$, smooth part 34;
8967 over $\{3, 7\}$, smooth part 147;
7598 over $\{2\}$, smooth part 2.

Smooth multiplica

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Choose y; imagine $y = 2^{40}$.
Define Q as set of primes $\leq$
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Know their factorizations.
Do linear algebra over $\mathbf{Z}/k$ on the exponent vectors.

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Smooth parts, new

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Algorithm:
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Smooth parts, new approach

Given nonzero $x_1, x_2, \ldots, x$
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Time typically $\le b(\lg b)^{2+o(}$
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Algorithm:
Compute $r = \prod_{p \in Q} p$.
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For each i separately:
Replace x_i by
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Smooth multiplicative dependencies

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Smooth parts, new approach

Given nonzero $x_1, x_2, \ldots, x_n \in \mathbf{Z}$ and finite set Q of primes:
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(2004 Franke Kleinjung Morain Wirth, in ECPP context)

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Or, to see if x_i is smooth,
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Smooth parts, new approach

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Slight variant (2004 Bernstein): Time always $\le b(\lg b)^{2+o(1)}$.

Compute smooth part of x_i as $\gcd\{x_i, (r \bmod x_i)^{2^k} \bmod x_i\}$ where $k = \lceil \lg \lg x_i \rceil$.

Subroutine: Computing gcd takes time $\le b(\lg b)^{2+o(1)}$. (1971 Schönhage; core idea: 1938 Lehmer; $b(\lg b)^{5+o(1)}$: 1971 Knuth)

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Minor problem: New algorithm
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Solution:
Feed the smooth numbers
to the old algorithm.
Very few smooth numbers,
so this is very fast.

Bottom line for cryptanalysis:
time per input number to
find and factor smooth numbers
has dropped by $(\lg b)^{1+o(1)}$.

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Is smooth the right question

After finding smooth numbe
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Throw away primes that app
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numbers with those primes;
repeat until stable.

Don't want *all* smooth num
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Minor problem: New algorithm finds the smooth numbers but doesn't factor them.

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Is smooth the right question?

After finding smooth numbers, do first step of linear algebra: Throw away primes that appear only once; throw away numbers with those primes; repeat until stable.

Don't want *all* smooth numbers. Want smooth numbers only if they are built from primes that divide the *other* numbers.

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Finds x_i iff all prin
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An alternate approach

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Finds x_i iff all primes in x_i are divisors of other x's.
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Given nonzero $x_1, x_2, \ldots, x_n \in \mathbf{Z}$:
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Compute $r = x_1 x_2 \cdots x_n$.
Compute (r/x_1) mod x_1, . . . ,
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An alternate approach

Given nonzero $x_1, x_2, \ldots, x_n \in \mathbf{Z}$:
Compute $r = x_1 x_2 \cdots x_n$.
Compute $(r/x_1) \bmod x_1$, ...,
$(r/x_n) \bmod x_n$.
For each i separately: see if
$((r/x_i) \bmod x_i)^{2^k} \bmod x_i = 0$
where $k = \lceil \lg \lg x_i \rceil$.

Finds x_i iff all primes in x_i
are divisors of other x's.
Time $\le b(\lg b)^{2+o(1)}$.

(2004 Bernstein)

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Exercise: Given 2^{23} RSA key how would you check for pri shared among those keys?

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