# Addition laws on elliptic curves

D. J. Bernstein
University of Illinois at Chicago

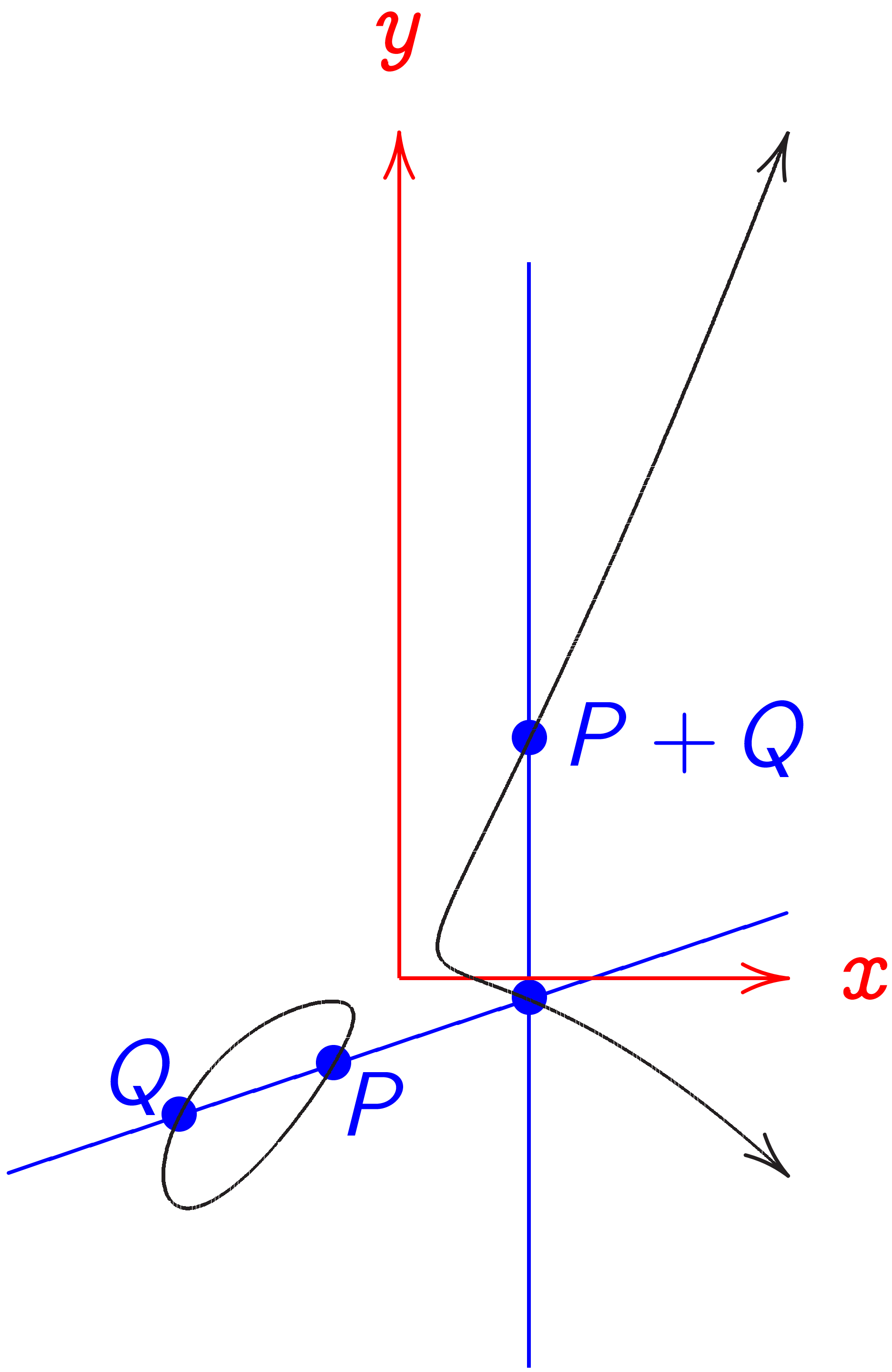
Joint work with:
Tanja Lange
Technische Universiteit Eindhoven

2007.01.10, 09:00 (yikes!), Leiden University, part of "Mathematics: Algorithms and Proofs" week at Lorentz Center:

Harold Edwards speaks on "Addition on elliptic curves."

Edwards

What we think when we hear "addition on elliptic curves":




Addition on $y^2 - 5xy = x^3 - 7$.

$$
\begin{aligned}
&\lambda = (y_2 - y_1)/(x_2 - x_1), \\
&x_3 = \lambda^2 - 5\lambda - x_1 - x_2, \\
&y_3 = 5x_3 - (y_1 + \lambda(x_3 - x_1)) \\
&\Rightarrow (x_1, y_1) + (x_2, y_2) = (x_3, y_3).
\end{aligned}
$$

$\lambda = (y_2 - y_1)/(x_2 - x_1)$,
$x_3 = \lambda^2 - 5\lambda - x_1 - x_2$,
$y_3 = 5x_3 - (y_1 + \lambda(x_3 - x_1))$
$\Rightarrow (x_1, y_1) + (x_2, y_2) = (x_3, y_3)$.
Oops, this requires $x_1 \neq x_2$.

$\lambda = (5y_1 + 3x_1^2)/(2y_1 - 5x_1)$,
$x_3 = \lambda^2 - 5\lambda - 2x_1$,
$y_3 = 5x_3 - (y_1 + \lambda(x_3 - x_1))$
$\Rightarrow (x_1, y_1) + (x_1, y_1) = (x_3, y_3)$.

$\lambda = (y_2 - y_1)/(x_2 - x_1)$,
$x_3 = \lambda^2 - 5\lambda - x_1 - x_2$,
$y_3 = 5x_3 - (y_1 + \lambda(x_3 - x_1))$
$\Rightarrow (x_1, y_1) + (x_2, y_2) = (x_3, y_3)$.
Oops, this requires $x_1 \neq x_2$.

$\lambda = (5y_1 + 3x_1^2)/(2y_1 - 5x_1)$,
$x_3 = \lambda^2 - 5\lambda - 2x_1$,
$y_3 = 5x_3 - (y_1 + \lambda(x_3 - x_1))$
$\Rightarrow (x_1, y_1) + (x_1, y_1) = (x_3, y_3)$.
Oops, this requires $2y_1 \neq 5x_1$.

$(x_1, y_1) + (x_1, 5x_1 - y_1) = \infty$.
$(x_1, y_1) + \infty = (x_1, y_1)$.
$\infty + (x_1, y_1) = (x_1, y_1)$.
$\infty + \infty = \infty$.

Despite 09:00,
despite Dutch trains,
we attend the talk.

Edwards says:

Euler–Gauss addition law
on $x^2 + y^2 = 1 - x^2y^2$ is
$(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$ with

$$x_3 = \frac{x_1y_2 + y_1x_2}{1 - x_1x_2y_1y_2},$$

$$y_3 = \frac{y_1y_2 - x_1x_2}{1 + x_1x_2y_1y_2}.$$


Euler

Gauss

Edwards, continued:

*Every* elliptic curve over $\overline{\mathbf{Q}}$ is birationally equivalent to $x^2 + y^2 = a^2(1 + x^2y^2)$ for some $a \in \overline{\mathbf{Q}} - \{0, \pm 1, \pm i\}$.

(Euler–Gauss curve $\equiv$ the "lemniscatic elliptic curve.")

Edwards, continued:

*Every* elliptic curve over $\overline{\mathbf{Q}}$
is birationally equivalent to
$x^2 + y^2 = a^2(1 + x^2y^2)$
for some $a \in \overline{\mathbf{Q}} - \{0, \pm 1, \pm i\}$.

(Euler–Gauss curve $\equiv$ the
"lemniscatic elliptic curve.")

$x^2 + y^2 = a^2(1 + x^2y^2)$ has
neutral element $(0, a)$, addition
$(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$ with

$$x_3 = \frac{x_1y_2 + y_1x_2}{a(1 + x_1x_2y_1y_2)},$$

$$y_3 = \frac{y_1y_2 - x_1x_2}{a(1 - x_1x_2y_1y_2)}.$$

Addition law is "unified":

$(x_1, y_1) + (x_1, y_1) = (x_3, y_3)$ with

$$x_3 = \frac{x_1 y_1 + y_1 x_1}{a(1 + x_1 x_1 y_1 y_1)},$$

$$y_3 = \frac{y_1 y_1 - x_1 x_1}{a(1 - x_1 x_1 y_1 y_1)}.$$

Have seen unification before.

e.g., 1986 Chudnovsky$^2$:

17**M** unified addition formulas

for $(S : C : D : Z)$ on Jacobi's

$S^2 + C^2 = Z^2$, $k^2 S^2 + D^2 = Z^2$.

Chudnovsky$^2$

Jacobi

2007.01.10, $\approx$ 09:30,
Bernstein–Lange:

Edwards addition law with standard projective $(X : Y : Z)$, standard Karatsuba optimization, common-subexp elimination:
$10\mathbf{M} + 1\mathbf{S} + 1\mathbf{A}$.
*Faster* than anything seen before!

**M**: field multiplication.
**S**: field squaring.
**A**: multiplication by $a$.

Karatsuba

Edwards paper: Bulletin AMS
**44** (2007), 393–422.

Many papers in 2007, 2008, 2009
have now used Edwards curves
to set speed records
for critical computations
in elliptic-curve cryptography.

Also new speed records
for ECM factorization: see
Lange's talk here on Saturday.

Also expect speedups in verifying
elliptic-curve primality proofs.

Back to B.–L., early 2007.

Edwards $x^2 + y^2 = a^2(1 + x^2y^2)$ doesn't *rationally* include Euler–Gauss $x^2 + y^2 = 1 - x^2y^2$.

Common generalization, presumably more curves over $\mathbf{Q}$, presumably more curves over $\mathbf{F}_q$:

$x^2 + y^2 = c^2(1 + dx^2y^2)$ has neutral element $(0, c)$, addition $(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$ with

$$x_3 = \frac{x_1y_2 + y_1x_2}{c(1 + dx_1x_2y_1y_2)},$$

$$y_3 = \frac{y_1y_2 - x_1x_2}{c(1 - dx_1x_2y_1y_2)}.$$

Convenient to take $c = 1$
for speed, simplicity.

Covers same set of curves
up to birational equivalence:
$(c, d) \equiv (1, dc^4)$.

$x^2 + y^2 = 1 + dx^2y^2$ has
neutral element $(0, 1)$, addition
$(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$ with

$$x_3 = \frac{x_1y_2 + y_1x_2}{1 + dx_1x_2y_1y_2},$$

$$y_3 = \frac{y_1y_2 - x_1x_2}{1 - dx_1x_2y_1y_2}.$$

Hmmm, does this really work?

Easiest way to check
the generalized addition law:
pull out the computer!

Pick a prime $p$; e.g. 47.
Pick curve param $d \in \mathbf{F}_p$.

Enumerate all affine points
$(x, y) \in \mathbf{F}_p \times \mathbf{F}_p$ satisfying
$x^2 + y^2 = 1 + dx^2y^2$.

Use generalized addition law
to make an addition table
for all pairs of points.
Check associativity etc.

Warning: Don't expect *complete* addition table.

Addition law works generically but can fail for some exceptional pairs of points.

Unified addition law works for generic additions and for generic doublings but can fail for some exceptional pairs of points.

Basic problem: Denominators $1 \pm dx_1x_2y_1y_2$ can be zero.

Even if we switched to projective coordinates, would expect addition law to fail for some points, producing $(0 : 0 : 0)$.

1995 Bosma–Lenstra theorem: "The smallest cardinality of a complete system of addition laws on $E$ equals two."

Bosma

Lenstra

Try $p = 47$, $d = 25$:
denominator $1 \pm dx_1x_2y_1y_2$
is nonzero for most points
$(x_1, y_1)$, $(x_2, y_2)$ on curve.
Edwards addition law is
associative whenever defined.

Try $p = 47$, $d = 25$:
denominator $1 \pm dx_1x_2y_1y_2$
is nonzero for most points
$(x_1, y_1)$, $(x_2, y_2)$ on curve.
Edwards addition law is
associative whenever defined.

Try $p = 47$, $d = -1$:
denominator $1 \pm dx_1x_2y_1y_2$
is nonzero for *all* points
$(x_1, y_1)$, $(x_2, y_2)$ on curve.
Addition law is a group law!

Try $p = 47$, $d = 25$:
denominator $1 \pm dx_1x_2y_1y_2$
is nonzero for most points
$(x_1, y_1)$, $(x_2, y_2)$ on curve.
Edwards addition law is
associative whenever defined.

Try $p = 47$, $d = -1$:
denominator $1 \pm dx_1x_2y_1y_2$
is nonzero for *all* points
$(x_1, y_1)$, $(x_2, y_2)$ on curve.
Addition law is a group law!

vs.


Z60T

2007 Bernstein–Lange completeness proof for all non-square $d$:

If $x_1^2 + y_1^2 = 1 + dx_1^2 y_1^2$
and $x_2^2 + y_2^2 = 1 + dx_2^2 y_2^2$
and $dx_1 x_2 y_1 y_2 = \pm 1$

2007 Bernstein–Lange
completeness proof
for all non-square $d$:

If $x_1^2 + y_1^2 = 1 + dx_1^2 y_1^2$
and $x_2^2 + y_2^2 = 1 + dx_2^2 y_2^2$
and $dx_1 x_2 y_1 y_2 = \pm 1$
then $dx_1^2 y_1^2 (x_2 + y_2)^2$
$= dx_1^2 y_1^2 (x_2^2 + y_2^2 + 2x_2 y_2)$
$= dx_1^2 y_1^2 (dx_2^2 y_2^2 + 1 + 2x_2 y_2)$

2007 Bernstein–Lange completeness proof for all non-square $d$:

If $x_1^2 + y_1^2 = 1 + dx_1^2y_1^2$
and $x_2^2 + y_2^2 = 1 + dx_2^2y_2^2$
and $dx_1x_2y_1y_2 = \pm 1$
then $dx_1^2y_1^2(x_2 + y_2)^2$
$= dx_1^2y_1^2(x_2^2 + y_2^2 + 2x_2y_2)$
$= dx_1^2y_1^2(dx_2^2y_2^2 + 1 + 2x_2y_2)$
$= d^2x_1^2y_1^2x_2^2y_2^2 + dx_1^2y_1^2 + 2dx_1^2y_1^2x_2y_2$

2007 Bernstein–Lange completeness proof for all non-square $d$:

If $x_1^2 + y_1^2 = 1 + dx_1^2 y_1^2$
and $x_2^2 + y_2^2 = 1 + dx_2^2 y_2^2$
and $dx_1 x_2 y_1 y_2 = \pm 1$
then $dx_1^2 y_1^2 (x_2 + y_2)^2$
$= dx_1^2 y_1^2 (x_2^2 + y_2^2 + 2x_2 y_2)$
$= dx_1^2 y_1^2 (dx_2^2 y_2^2 + 1 + 2x_2 y_2)$
$= d^2 x_1^2 y_1^2 x_2^2 y_2^2 + dx_1^2 y_1^2 + 2dx_1^2 y_1^2 x_2 y_2$
$= 1 + dx_1^2 y_1^2 \pm 2x_1 y_1$

2007 Bernstein–Lange
completeness proof
for all non-square $d$:

If $x_1^2 + y_1^2 = 1 + dx_1^2 y_1^2$
and $x_2^2 + y_2^2 = 1 + dx_2^2 y_2^2$
and $dx_1 x_2 y_1 y_2 = \pm 1$
then $dx_1^2 y_1^2 (x_2 + y_2)^2$
$= dx_1^2 y_1^2 (x_2^2 + y_2^2 + 2x_2 y_2)$
$= dx_1^2 y_1^2 (dx_2^2 y_2^2 + 1 + 2x_2 y_2)$
$= d^2 x_1^2 y_1^2 x_2^2 y_2^2 + dx_1^2 y_1^2 + 2dx_1^2 y_1^2 x_2 y_2$
$= 1 + dx_1^2 y_1^2 \pm 2x_1 y_1$
$= x_1^2 + y_1^2 \pm 2x_1 y_1 = (x_1 \pm y_1)^2$.

2007 Bernstein–Lange completeness proof for all non-square $d$:

If $x_1^2 + y_1^2 = 1 + dx_1^2y_1^2$
and $x_2^2 + y_2^2 = 1 + dx_2^2y_2^2$
and $dx_1x_2y_1y_2 = \pm 1$
then $dx_1^2y_1^2(x_2 + y_2)^2$
$= dx_1^2y_1^2(x_2^2 + y_2^2 + 2x_2y_2)$
$= dx_1^2y_1^2(dx_2^2y_2^2 + 1 + 2x_2y_2)$
$= d^2x_1^2y_1^2x_2^2y_2^2 + dx_1^2y_1^2 + 2dx_1^2y_1^2x_2y_2$
$= 1 + dx_1^2y_1^2 \pm 2x_1y_1$
$= x_1^2 + y_1^2 \pm 2x_1y_1 = (x_1 \pm y_1)^2$.

Have $x_2 + y_2 \neq 0$ or $x_2 - y_2 \neq 0$; either way $d$ is a square. Q.E.D.

1995 Bosma–Lenstra theorem:
"The smallest cardinality of a complete system of addition laws on $E$ equals two."

1995 Bosma–Lenstra theorem:
"The smallest cardinality of a complete system of addition laws on $E$ equals two." ... meaning:
Any addition formula
for a Weierstrass curve $E$
in projective coordinates
must have exceptional cases
in $E(\overline{k}) \times E(\overline{k})$, where
$\overline{k}$ = algebraic closure of $k$.

1995 Bosma–Lenstra theorem: “The smallest cardinality of a complete system of addition laws on $E$ equals two.” ... meaning:
Any addition formula
for a Weierstrass curve $E$
in projective coordinates
must have exceptional cases
in $E(\overline{k}) \times E(\overline{k})$, where
$\overline{k}$ = algebraic closure of $k$.

Edwards addition formula has exceptional cases for $E(\overline{k})$
... but not for $E(k)$.
We do computations in $E(k)$.

Summary: Assume $k$ field; $2 \neq 0$ in $k$; non-square $d \in k$. Then $\{(x, y) \in k \times k :$

$$x^2 + y^2 = 1 + dx^2y^2\}$$

is a commutative group with $(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$ defined by Edwards addition law:

$$x_3 = \frac{x_1y_2 + y_1x_2}{1 + dx_1x_2y_1y_2},$$

$$y_3 = \frac{y_1y_2 - x_1x_2}{1 - dx_1x_2y_1y_2}.$$

Terminology: "Edwards curves" allow arbitrary $d \in k^*$; $d = c^4$ are "original Edwards curves"; non-square $d$ are "complete."

$d = 0$: "the clock group."
$x^2 + y^2 = 1$, parametrized
by $(x, y) = (\sin, \cos)$.

Gauss parametrized
$x^2 + y^2 = 1 - x^2y^2$ by
$(x, y) = ($"lemn sin", "lemn cos"$)$.

Abel, Jacobi "sn, cn, dn"
handle all elliptic curves,
but (sn, cn) does *not*
specialize to (lemn sin, lemn cos).
Bad generalization of (sin, cos).

Edwards $x$ is sn;
Edwards $y$ is cn/dn.
Theta view: see Edwards paper.

Every elliptic curve over $k$
with a point of order 4
is birationally equivalent
to an Edwards curve.

Unique order-2 point $\Rightarrow$ complete.
Convenient for implementors:
no need to worry about
accidentally bumping into
exceptional inputs.

Particularly nice for cryptography:
no need to worry about
attackers manufacturing
exceptional inputs,
hearing case distinctions, etc.

What about elliptic curves
without points of order 4?

What about elliptic curves
over binary fields?

Continuing project (B.–L.):
For *every* elliptic curve $E$,
find complete addition law for $E$
with best possible speeds.

Complete laws are useful
even if slower than Edwards!

2008 B.–Birkner–L.–Peters:
"twisted Edwards curves"
$ax^2 + y^2 = 1 + dx^2y^2$
cover all Montgomery curves.

Almost as fast as $a = 1$;
brings Edwards speed
to larger class of curves.

2008 B.–B.–Joye–L.–P.:
every elliptic curve over $\mathbf{F}_p$
where 4 divides group order
is (1 or 2)-isogenous
to a twisted Edwards curve.

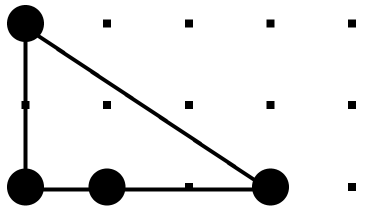
Statistics for many $p \in 1 + 4\mathbf{Z}$,
$\approx$ number of pairs $(j(E), \#E)$:

| Curves | total | odd | 2odd | 4odd | 8odd |
|---|---|---|---|---|---|
| orig | $\frac{1}{24}p$ | 0 | 0 | 0 | 0 |
| compl | $\frac{1}{2}p$ | 0 | 0 | $\frac{1}{4}p$ | $\frac{1}{8}p$ |
| Ed | $\frac{2}{3}p$ | 0 | 0 | $\frac{1}{4}p$ | $\frac{3}{16}p$ |
| twist | $\frac{5}{6}p$ | 0 | 0 | $\frac{5}{12}p$ | $\frac{3}{16}p$ |
| $4\mathbf{Z}$ | $\frac{5}{6}p$ | 0 | 0 | $\frac{5}{12}p$ | $\frac{3}{16}p$ |
| all | $2p$ | $\frac{2}{3}p$ | $\frac{1}{2}p$ | $\frac{5}{12}p$ | $\frac{3}{16}p$ |

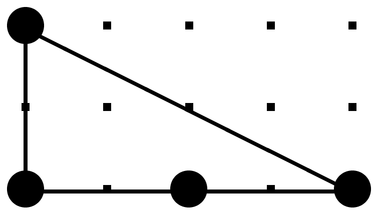
Different statistics for $3 + 4\mathbf{Z}$.

Bad news:
complete twisted Edwards
$\equiv$ complete Edwards!

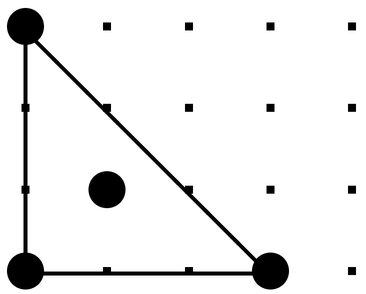
# Some Newton polygons



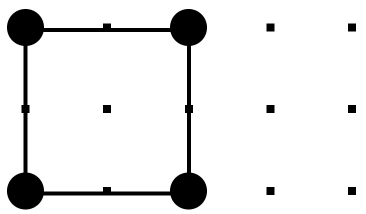
Short Weierstrass



Jacobi quartic



Hessian



Edwards

1893 Baker: genus is generically number of interior points.

2000 Poonen–Rodriguez-Villegas classified genus-1 polygons.

How to generalize Edwards?

Design decision: want quadratic in $x$ and in $y$.

Design decision: want $x \leftrightarrow y$ symmetry.

$$\begin{array}{ccc} d_{20} & d_{21} & d_{22} \\ \\ d_{10} & d_{11} & d_{21} \\ \\ d_{00} & d_{10} & d_{20} \end{array}$$

Curve shape $d_{00} + d_{10}(x + y) + d_{11}xy + d_{20}(x^2 + y^2) + d_{21}xy(x + y) + d_{22}x^2y^2 = 0$.

Suppose that $d_{22} = 0$:

$$\begin{array}{ccc} d_{20} & d_{21} & \cdot \\ d_{10} & d_{11} & d_{21} \\ d_{00} & d_{10} & d_{20} \end{array}$$

Genus $1 \Rightarrow (1,1)$ is an interior point $\Rightarrow d_{21} \neq 0$.

Homogenize:

$$d_{00}Z^3 + d_{10}(X+Y)Z^2 + d_{11}XYZ + d_{20}(X^2+Y^2)Z + d_{21}XY(X+Y) = 0.$$

Points at $\infty$ are $(X : Y : 0)$ with $d_{21}XY(X+Y) = 0$: i.e., $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(1 : -1 : 0)$.

Study $(1 : 0 : 0)$ by setting $y = Y/X$, $z = Z/X$ in homogeneous curve equation:

$$d_{00}z^3 + d_{10}(1+y)z^2 + d_{11}yz + d_{20}(1+y^2)z + d_{21}y(1+y) = 0.$$

Nonzero coefficient of $y$ so $(1 : 0 : 0)$ is nonsingular. Addition law cannot be complete (unless $k$ is tiny).

So we require $d_{22} \neq 0$.

Points at $\infty$ are $(X : Y : 0)$ with $d_{22}X^2Y^2 = 0$: i.e., $(1 : 0 : 0)$, $(0 : 1 : 0)$.

Study $(1 : 0 : 0)$ again:
$d_{00}z^4 + d_{10}(1 + y)z^3 + d_{11}yz^2 + d_{20}(1 + y^2)z^2 + d_{21}y(1 + y)z + d_{22}y^2 = 0$.

Coefficients of $1, y, z$ are $0$ so $(1 : 0 : 0)$ is singular.

Put $y = uz$, divide by $z^2$ to blow up singularity:

$$d_{00}z^2 + d_{10}(1 + uz)z + d_{11}uz + d_{20}(1 + u^2z^2) + d_{21}u(1 + uz) + d_{22}u^2 = 0.$$

Substitute $z = 0$ to find points above singularity: $d_{20} + d_{21}u + d_{22}u^2 = 0.$

We require the quadratic $d_{20} + d_{21}u + d_{22}u^2$ to be irreducible in $k$.
Special case: complete Edwards, $1 - du^2$ irreducible in $k$.

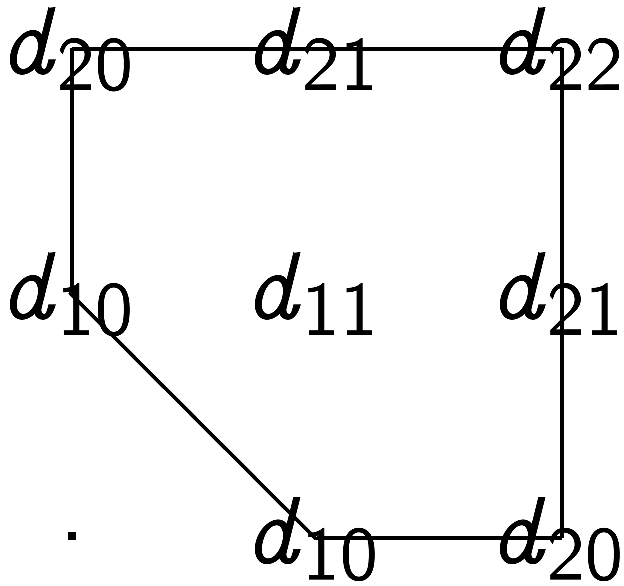
In particular $d_{20} \neq 0$:

$$\begin{array}{ccc} d_{20} & d_{21} & d_{22} \\ d_{10} & d_{11} & d_{21} \\ d_{00} & d_{10} & d_{20} \end{array}$$

Design decision: Explore
a deviation from Edwards.
Choose neutral element $(0, 0)$.
$d_{00} = 0$; $d_{10} \neq 0$.

Can vary neutral element.
Warning: bad choice can produce
surprisingly expensive negation.

Now have a Newton polygon
for generalized Edwards curves:




By scaling $x, y$
and scaling curve equation
can limit $d_{10}, d_{11}, d_{20}, d_{21}, d_{22}$
to three degrees of freedom.

2008 B.–L.–Rezaeian Farashahi: complete addition law for "binary Edwards curves" $d_1(x+y)+d_2(x^2+y^2) = (x+x^2)(y+y^2)$.

Covers all ordinary elliptic curves over $\mathbf{F}_{2^n}$ for $n \ge 3$.

Also surprisingly fast, especially if $d_1 = d_2$.

2008 B.–L.–Rezaeian Farashahi:
complete addition law for
“binary Edwards curves”
$d_1(x+y)+d_2(x^2+y^2) = (x+x^2)(y+y^2)$.
Covers all ordinary elliptic curves over $\mathbf{F}_{2^n}$ for $n \geq 3$.
Also surprisingly fast, especially if $d_1 = d_2$.

2009 B.–L.:
complete addition law for another specialization covering all the “NIST curves” over *non-binary* fields.

Consider, e.g., the curve

$$x^2 + y^2 = x + y + txy + dx^2y^2$$

with $d = -1$ and

$$t = \begin{array}{r} 78751018041117252545420999954 \\ 76717646453854506081463020284 \\ 13956511758592017 99 \end{array}$$

over $\mathbf{F}_p$ where $p = 2^{256} - 2^{224} + 2^{192} + 2^{96} - 1$.

Note: $d$ is non-square in $\mathbf{F}_p$.

Birationally equivalent to standard "NIST P-256" curve $v^2 = u^3 - 3u + a_6$ where

$$a_6 = \begin{array}{r} 41058363725152142129326129780 \\ 04726840911444101599372555483. \\ 525631403946740 1291 \end{array}$$

An addition law for
$x^2 + y^2 = x + y + txy + dx^2y^2$,
complete if $d$ is not a square:

$$x_3 = \frac{\begin{array}{l} x_1 + x_2 + (t-2)x_1x_2 + \\ (x_1 - y_1)(x_2 - y_2) + \\ dx_1^2(x_2y_1 + x_2y_2 - y_1y_2) \end{array}}{\begin{array}{l} 1 - 2dx_1x_2y_2 - \\ dx_1^2(x_2 + y_2 + (t-2)x_2y_2) \end{array}};$$

$$y_3 = \frac{\begin{array}{l} y_1 + y_2 + (t-2)y_1y_2 + \\ (y_1 - x_1)(y_2 - x_2) + \\ dy_1^2(y_2x_1 + y_2x_2 - x_1x_2) \end{array}}{\begin{array}{l} 1 - 2dy_1y_2x_2 - \\ dy_1^2(y_2 + x_2 + (t-2)y_2x_2) \end{array}}.$$

Note on computing addition laws:
An easy Magma script uses Riemann–Roch to find addition law given a curve shape.

Are those laws nice? No!
Find lower-degree laws by Monagan–Pearce algorithm, ISSAC 2006; or by evaluation at random points on random curves.

Are those laws complete? No!
But always seems easy to find complete addition laws among low-degree laws where denominator constant term $\neq 0$.

Birational equivalence from
$x^2+y^2=x+y+txy+dx^2y^2$ to
$$v^2-(t+2)uv+dv= u^3-(t+2)u^2-du+(t+2)d$$
i.e. $v^2-(t+2)uv+dv= (u^2-d)(u-(t+2))$:

$$u=(dxy+t+2)/(x+y);$$
$$v=\frac{((t+2)^2-d)x}{(t+2)xy+x+y}.$$

Assuming $t+2$ square, $d$ not:
only exceptional point is
$(0,0)$, mapping to $\infty$.

Inverse: $x=v/(u^2-d)$;
$y=((t+2)u-v-d)/(u^2-d)$.

# Completeness

$$x_3 = \frac{x_1 + x_2 + (t-2)x_1x_2 + (x_1 - y_1)(x_2 - y_2) + dx_1^2(x_2y_1 + x_2y_2 - y_1y_2)}{1 - 2dx_1x_2y_2 - dx_1^2(x_2 + y_2 + (t-2)x_2y_2)};$$

$$y_3 = \frac{y_1 + y_2 + (t-2)y_1y_2 + (y_1 - x_1)(y_2 - x_2) + dy_1^2(y_2x_1 + y_2x_2 - x_1x_2)}{1 - 2dy_1y_2x_2 - dy_1^2(y_2 + x_2 + (t-2)y_2x_2)}.$$

Can denominators be 0?

Only if $d$ is a square!

Theorem: Assume that

$k$ is a field with $2 \neq 0$;

$d, t, x_1, y_1, x_2, y_2 \in k$;

$d$ is not a square in $k$;

$27d \neq (2-t)^3$;

$x_1^2 + y_1^2 = x_1 + y_1 + t x_1 y_1 + d x_1^2 y_1^2$;

$x_2^2 + y_2^2 = x_2 + y_2 + t x_2 y_2 + d x_2^2 y_2^2$.

Then $1 - 2d x_1 x_2 y_2 -$

$d x_1^2 (x_2 + y_2 + (t-2) x_2 y_2) \neq 0$.

Only if $d$ is a square!

Theorem: Assume that
$k$ is a field with $2 \neq 0$;
$d, t, x_1, y_1, x_2, y_2 \in k$;
$d$ is not a square in $k$;
$27d \neq (2-t)^3$;
$x_1^2+y_1^2 = x_1+y_1+tx_1y_1+dx_1^2y_1^2$;
$x_2^2+y_2^2 = x_2+y_2+tx_2y_2+dx_2^2y_2^2$.
Then $1 - 2dx_1x_2y_2 -$
$dx_1^2(x_2 + y_2 + (t-2)x_2y_2) \neq 0$.

By $x \leftrightarrow y$ symmetry
also $1 - 2dy_1y_2x_2 -$
$dy_1^2(y_2 + x_2 + (t-2)y_2x_2) \neq 0$.

Proof: Suppose that

$$1 - 2dx_1x_2y_2 - dx_1^2(x_2 + y_2 + (t-2)x_2y_2) = 0.$$

Proof: Suppose that

$$1 - 2dx_1x_2y_2 - dx_1^2(x_2 + y_2 + (t-2)x_2y_2) = 0.$$

Note that $x_1 \neq 0$.

Proof: Suppose that

$1 - 2dx_1x_2y_2 -$
$dx_1^2(x_2 + y_2 + (t-2)x_2y_2) = 0.$

Note that $x_1 \neq 0$.

Use curve equation$_2$ to see that
$(1 - dx_1x_2y_2)^2 = dx_1^2(x_2 - y_2)^2.$

Proof: Suppose that

$1 - 2dx_1x_2y_2 -$
$dx_1^2(x_2 + y_2 + (t-2)x_2y_2) = 0.$

Note that $x_1 \neq 0$.

Use curve equation$_2$ to see that
$(1 - dx_1x_2y_2)^2 = dx_1^2(x_2 - y_2)^2.$

By hypothesis $d$ is non-square
so $x_1^2(x_2 - y_2)^2 = 0$
and $(1 - dx_1x_2y_2)^2 = 0.$

Proof: Suppose that

$1 - 2dx_1x_2y_2 -$
$dx_1^2(x_2 + y_2 + (t - 2)x_2y_2) = 0.$

Note that $x_1 \neq 0$.

Use curve equation$_2$ to see that
$(1 - dx_1x_2y_2)^2 = dx_1^2(x_2 - y_2)^2$.

By hypothesis $d$ is non-square
so $x_1^2(x_2 - y_2)^2 = 0$
and $(1 - dx_1x_2y_2)^2 = 0$.

Hence $x_2 = y_2$ and $1 = dx_1x_2y_2$.

Curve equation$_1$ times $1/x_1^2$:

$1 + y_1^2/x_1^2 =$

$1/x_1 + y_1(1/x_1^2 + t/x_1) + dy_1^2$.

Curve equation$_1$ times $1/x_1^2$:

$1 + y_1^2/x_1^2 =$
$1/x_1 + y_1(1/x_1^2 + t/x_1) + dy_1^2$.

Substitute $1/x_1 = dx_2^2$:

$1 + d^2y_1^2x_2^4 =$
$dx_2^2 + dy_1(dx_2^4 + x_2^2t) + dy_1^2$.

Curve equation$_1$ times $1/x_1^2$:

$1+y_1^2/x_1^2 =$

$1/x_1+y_1(1/x_1^2+t/x_1)+dy_1^2$.

Substitute $1/x_1=dx_2^2$:

$1+d^2y_1^2x_2^4 =$

$dx_2^2+dy_1(dx_2^4+x_2^2t)+dy_1^2$.

Substitute $2x_2^2=2x_2+tx_2^2+dx_2^4$:

$(1-dy_1x_2^2)^2=d(x_2-y_1)^2$.

Curve equation$_1$ times $1/x_1^2$:
$1 + y_1^2/x_1^2 =$
$1/x_1 + y_1(1/x_1^2 + t/x_1) + dy_1^2.$

Substitute $1/x_1 = dx_2^2$:
$1 + d^2y_1^2x_2^4 =$
$dx_2^2 + dy_1(dx_2^4 + x_2^2t) + dy_1^2.$

Substitute $2x_2^2 = 2x_2 + tx_2^2 + dx_2^4$:
$(1 - dy_1x_2^2)^2 = d(x_2 - y_1)^2.$

Thus $x_2 = y_1$ and $1 = dy_1x_2^2$.
Hence $1 = dx_2^3$.

Curve equation$_1$ times $1/x_1^2$:
$1 + y_1^2/x_1^2 =$
$1/x_1 + y_1(1/x_1^2 + t/x_1) + dy_1^2$.

Substitute $1/x_1 = dx_2^2$:
$1 + d^2y_1^2x_2^4 =$
$dx_2^2 + dy_1(dx_2^4 + x_2^2t) + dy_1^2$.

Substitute $2x_2^2 = 2x_2 + tx_2^2 + dx_2^4$:
$(1 - dy_1x_2^2)^2 = d(x_2 - y_1)^2$.

Thus $x_2 = y_1$ and $1 = dy_1x_2^2$.
Hence $1 = dx_2^3$.

Now $2x_2^2 = 2x_2 + tx_2^2 + x_2$
so $3 = (2-t)x_2$ so $27d = (2-t)^3$.
Contradiction.

## What's next?

Make the mathematicians happy:
*Prove* that all curves
are covered; should be easy
using Weil and rational param.

Make the computer happy:
Find *faster* complete laws.

Latest news, B.–Kohel–L.:
Have complete addition law
for twisted Hessian curves
$ax^3 + y^3 + 1 = 3dxy$
when $a$ is non-cube.
Close in speed to Edwards
and covers different curves.