Proving tight security for Rabin-Williams signatures

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Public-key signatures

1976 Diffie Hellman:
Public-key signatures
would allow verification by
anyone, not just signer. Cool!
Can we build a signature system?

1977 Rivest Shamir Adleman:
verify $s^e - m \in pq\mathbf{Z}$.
Public (pq, e) with big random e;
message m; signature $s \in \mathbf{Z}/pq$.

1977 (and 1978) RSA was slow:
many mults in eth powering.
Even worse, horribly insecure:
e.g. forge $(m, s) = (1, 1)$.

1979 Rabin: $s^2 - H(m) \in pq\mathbf{Z}$.
Standard H. Public pq.
Message m. Signature s.
Fast; conjecturally secure; but can sign only $\approx 1/4$ of all m's.

1979 Rabin: $s^2 - H(r, m) \in pq\mathbf{Z}$.
Signature (r, s) instead of s.
Signer tries random r's, on average ≈ 4 times.

1980 Williams: $efs^2 - \cdots \in pq\mathbf{Z}$;
$e \in \{-1, 1\}$; $f \in \{1, 2\}$.
Each $h \in \mathbf{Z}/pq$ is efs^2 for exactly 4 vectors (e, f, s) if $p \in 3 + 8\mathbf{Z}$, $q \in 7 + 8\mathbf{Z}$.

Subsequent RW variations:

- Eliminate Euclid, Jacobi.
- Expand s for faster verification.
- Compress s to $1/2$ size.
- Compress pq to $1/3$ size.
- Compress ("recover") m via H.

Many other signature systems (e.g. elliptic-curve Schnorr), but RW family still holds verification speed records.

RW is the best choice for verification-heavy applications: e.g., Internet DNS security.

Attacks against RW

Attacker sees public key pq and many vectors (m, e, f, r, s) legitimately signed under that key.

Attacker forges (m', e', f', r', s') with $e' \in \{-1, 1\}$, $f' \in \{1, 2\}$, r' of standard length, $s' \in \mathbf{Z}/pq$.

Forgery is **successful** if $e'f'(s')^2 - H(r', m') \in pq\mathbf{Z}$ and m' wasn't legitimately signed.

Fundamental security question: What's maximum Pr[A succeeds] among all feasible attacks A?

Maybe answer depends on how messages are generated.

We want Pr[A succeeds] small for *all* message generators and all feasible attacks A.

Different users have different types of message generators, communication between attacker and message generator, etc. Would be painful to analyze each generator separately.

Similarly, would be painful to limit set of messages.

Attack 1: Blind H inversion.

Attacker chooses e', f', s', chooses $h \in e'f'(s')^2 + pq\mathbf{Z}$, guesses uniform random r', m' until finding $H(r', m') = h$, forges (m', e', f', r', s').

Obstacle to success of attack: What's chance of finding $H(r', m') = h$ after a feasible number of guesses? Conjecturally negligible for every popular H.

Attack 2: Blind collision search.

Attacker guesses r', m', r'', m with $m' \neq m$ and $H(r', m') = H(r'', m)$. Message generator gives m to signer: (m, e, f, r, s). Attacker forges (m', e, f, r', s).

Forgery succeeds if $r = r''$: $H(r, m) = H(r'', m) = H(r', m')$. Good chance if r is short.

Same obstacle as before: Feasible number of guesses has conjecturally negligible chance of finding collision in H.

Attack 3: MD5 collision search.

Was popular last decade
to build $H(x) = G(\text{MD5}(x))$
for standard function G.
Assume this shape of H.

Feasible calculation,
highly non-uniform guessing,
finds collisions in MD5.
(2004 Wang Feng Lai Yu)

Thus obtain collision in H.
Forge as in attack 2.
Good chance of success
if r is short. Feasible attack!

One reaction to this attack:
MD5 was a bad design.
Change choice of H.
Collisions conjecturally infeasible
for many popular H's.

Another reaction (1979 Rabin,
1989 Schnorr, et al.):
Standardize 256-bit r.
Negligible chance of $r = r''$.
Inversions conjecturally infeasible
for many popular H's.

Is second reaction better?
Long r is clear disadvantage.
Maybe outweighed by faster H?

Attack 4: Factorization by NFS.

Attacker hires computational number theorist to factor n using the number-field sieve. Attacker chooses m', signs m' same way as legitimate signer.

1978 RSA: "We recommend using 100-digit (decimal) prime numbers p and q, so that n has 200 digits."

2005 Bahr Boehm Franke Kleinjung: "We have factored RSA200 by GNFS."

Attack 5: Leak detection.

Signer has many choices of signature for m: 2^B choices of B-bit r, and then 4 choices of (e, f, s).

Imagine idiotic signer making successive bits of p visible to attacker by, e.g., copying them into bits of r or into Jacobi symbol of s mod pq.

Evidently security depends on choice of signing algorithm.

Many more attacks in literature.

Many (most?) of the attacks
are H-**generic**:
attack works for every function H
(or a large fraction of H's)
if signer, attacker, verifier
use an oracle for H.

It's quite embarrassing
for a system to be broken
by an H-generic attack
faster than factorization!

Example: Signing-leak attacks
are H-generic, embarrassing.

1987 Fiat Shamir:
Here's a signature system
where embarrassment is limited.
Can convert H-generic attack
into factorization algorithm
with only polynomial loss of
efficiency and effectiveness.

1996 Bellare Rogaway:
Here's a signature system
immune to embarrassment.
Can convert H-generic attack
into factorization algorithm
with almost negligible loss of
efficiency and effectiveness.

Many subsequent systems and "reductions in the random-oracle model." Confusing terminology.

Common flaws in the theorems:
- Reductions aren't very tight.
- Tightness isn't quantified.
- Proofs have gaps, errors.
- The theorems don't apply to the fastest systems.

The point of this talk:
We can do better! Now have very tight proofs for some state-of-the-art RW variants.
(most recently 2006 Bernstein)

Three state-of-the-art systems

“Fixed 0-bit unstructured RW” is immune to embarrassment.

“0-bit”: 0 bits in r (despite 2002 Coron theorem that “FDH can’t be tight”).

“Unstructured”: Signer’s choice of (e, f, s) is uniform random, independent of (p, q).

“Fixed”: Given same m again, signer chooses same signature.

For comparison, easily break “variable 0-bit unstructured RW.”

“Fixed” needs memory for signatures of old m’s.

But, without memory, can produce indistinguishable results, assuming standard conjectures in secret-key cryptography: signer generates “random” bits by hashing m together with a secret independent of p, q. (1997 Barwood, 1997 Wigley)

Can hash m using Poly1305 or forthcoming MAC1071; just a few cycles per byte. Scramble output with Salsa20.

“Fixed 1-bit principal RW”
is immune to embarrassment.

“1-bit”: uniform random bit r,
independent of (p, q).

“Principal”: Signer chooses
$e = 1$ when there’s a choice;
$f = 1$ when there’s a choice;
and the unique $s \in \mathbf{Z}/pq$
that’s a square in $\mathbf{Z}/pq$.

(Same idea as 2003 Katz Wang
reduction for fixed 1-bit RSA etc.,
but need generalization beyond
“claw-free permutation pairs.”)

“Fixed 128-bit |principal| RW” is immune to embarrassment.

“128-bit”: uniform random 128-bit r, independent of (p, q).

“|Principal|”: Signer chooses $e = 1$ when there’s a choice; $f = 1$ when there’s a choice; and $s \in \{0, 1, \ldots, (pq - 1)/2\}$ with s or $-s$ square in $\mathbf{Z}/pq$.

Implementation note: Can continue to avoid Euclid, Jacobi.

Blind attacks

Consider algorithm A_1 that, given $pq \in \{2^{2048}, \ldots, 2^{2049} - 1\}$ and h', computes (e', f', s').

How large is $\Pr[e'f'(s')^2 = h']$ for uniform random 2048-bit h'?

Build factorization algorithm A_0:
choose uniform random (e, f, s);
compute $h' = efs^2$;
start over if $h' \geq 2^{2048}$;
compute $(e', f', s') = A_1(pq, h')$;
compute $\gcd\{pq, s' - s\}$.
Comparable efficiency to A_1.

Define A_1 as **successful**
if $e'f'(s')^2 = h'$.

If A_1 is always successful
then $e'f'(s')^2 = efs^2$;
s', s are coprime to pq
with probability $1 - \epsilon$, tiny ϵ;
s', s are independent square roots;
so $\gcd\{pq, s' - s\} \in \{p, q\}$
with probability $\geq (1 - \epsilon)/2$.

More generally:
If $\Pr[A_1 \text{ succeeds}] = \alpha$ for
uniform random 2048-bit h' then
$\Pr[A_0 \text{ factors } pq] \geq \alpha(1 - \epsilon)/2$.

Seeing a signature

Consider algorithm A_2 that, given h, e, f, s, h', pq with $h = efs^2$, computes (e', f', s').

How large is $\Pr[e'f'(s')^2 = h']$ for independent uniform random 2048-bit h, h'?

Three versions of question:

1. Unstructured (e, f, s).
2. Principal (e, f, s).
3. $|$Principal$|$ (e, f, s).

Analogy: Attack sees signature of m, forges signature of $m' \neq m$.

Intuition: A_2 learns nothing from seeing h, e, f, s.

Formalization:
Simulated signer, given pq, generates random (h, e, f, s) with exactly the right distribution.

Thus can build A_1 from A_2.

A_1, given pq, h',
generates h, e, f, s;
runs A_2 with h, e, f, s, h', pq.
$\Pr[A_2 \, \text{succeeds}] = \Pr[A_1 \, \text{succeeds}]$.

How to generate h, e, f, s?
How does simulated signer work?

For unstructured:
Generate uniform random e, f, s;
compute $h = efs^2$;
start over if $h \geq 2^{2048}$.

For principal:
Generate uniform random e, f, x;
compute $s = x^2$;
tweak e, f if $\gcd\{x, pq\} > 1$;
compute $h = efs^2$;
start over if $h \geq 2^{2048}$.

For $|$principal$|$: Similar.

Seeing many signatures

Consider A_3 that, given pq, h', $(h_1, e_1, f_1, s_1), \ldots, (h_q, e_q, f_q, s_q)$ with each $h_i = e_i f_i s_i^2$, computes (e', f', s').

How large is $\Pr[e' f' (s')^2 = h']$ for independent uniform random $h_1, \ldots, h_q, h'$? Again three versions of question.

Analogy: Attack sees signatures of distinct $m_1, \ldots, m_q$, forges signature of $m' \notin \{m_1, \ldots, m_q\}$.

Intuition: A_3 learns nothing from seeing h_i, e_i, f_i, s_i.

Formalize exactly as before. Build A_1 from A_3 by generating h_i, e_i, f_i, s_i using same simulated signer.

Warning regarding distinctness: Reasonable to vary problem, forcing $h_i = h_j$ for various (i, j); analogous to attacker forcing repetitions in $m_1, \ldots, m_q$. Same simulated signer is fine for fixed signatures but not for variable signatures.

Hashing first

The "FDH" case ($B = 0$, no r):

Consider algorithm A_4 that
is given $pq, h_1, h_2, \ldots, h_{q+1}$;
selects i, sees e_i, f_i, s_i;
repeats for any number of i's;
computes i', e', f', s'.

Algorithm is **successful**
if $h_{i'} = e' f' (s')^2$ and i' is new.
How large is Pr[A_4 succeeds]?

Analogy: Attack chooses
messages to feed to legitimate
signer *after* inspecting hashes.

Conventional treatment of FDH
(1996 Bellare Rogaway, etc.):

Easily build A_3 from A_4
by guessing i' in advance.
Guess is correct
with probability $1/(q+1)$.

Can increase probability
from $1/\#\{\text{hash queries}\}$
to $\Theta(1/\#\{\text{signing queries}\})$.
(2000 Coron)

"We show . . . it is not possible to
further improve the security proof
of FDH." (2002 Coron)

New treatment (2006 Bernstein):

Easily build A_0 from A_4 for unstructured signatures. Tight! No guessing required. Warning: construction fails for principal, $|$principal$|$, etc.

For each $i \in \{1, \ldots, q+1\}$ choose independent uniform random (e_i, f_i, s_i). No need to distinguish i' from the signing queries.

2002 Coron theorem assumes "unique signature." Signatures here aren't "unique."

The non-“FDH” case, $B \geq 1$:

Consider algorithm A_4 that
is given random access to pq,
$h_1(0), \ldots, h_1(2^B - 1)$,
$h_2(0), \ldots, h_2(2^B - 1), \ldots$,
$h_{q+1}(0), \ldots, h_{q+1}(2^B - 1)$;
selects i, sees $r_i, h_i(r_i), e_i, f_i, s_i$;
repeats for any number of i's;
computes i', e', f', r', s'.

Algorithm is **successful** if
$h_{i'}(r') = e' f' (s')^2$ and i' is new.

Analogy: $h_i(r) = H(r, m_i)$,
distinct $m_1, \ldots, m_{q+1}$.

Unstructured: as for $B = 0$.
What about principal etc.?

Resort to 2003 Katz Wang.
Katz-Wang theorem is limited to
"claw-free permutation pairs"
but idea also works for RW.

Build $h_i(r)$ for $r \neq r_i$
using unstructured simulator.
Build $h_i(r)$ for $r = r_i$
using principal simulator.
If i' new then $r', r_{i'}$ independent
so $\Pr[r' \neq r_{i'}] = 1 - 1/2^B$.

Probability loss is 2 for $B = 1$;
converges rapidly to 1 as $B \to \infty$.

The final reduction

Consider attack A_5 that
is given H oracle and pq;
selects m, sees signature;
repeats for any number of m's;
computes (m', e', f', r', s').

Easily convert into A_4,
thanks to fixed signatures.
$\Pr[A_5 \text{ succeeds}] = \Pr[A_4 \text{ succeeds}]$
assuming uniform random H.

Credit for exposition:
I stole A_4 shape from
2004 Koblitz Menezes ("RSA1").

The “random-oracle” debate

“These proofs are required!
Security must be proven!
Cryptosystems without theorems
are bad cryptosystems!”

“No! These proofs are useless!
Some attacks aren’t generic!”

My view: Insisting on proofs
exposes many security problems,
weeds out many awful systems,
saves time for cryptographers.
Reconsider this insistence
if it sacrifices system speed;
but today there’s no conflict.