Efficient arithmetic in finite fields

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Some examples of finite fields:

$\mathbf{Z}/(2^{255} - 19)$.
$(\mathbf{Z}/(2^{61} - 1))[t]/(t^5 - 3)$.
$(\mathbf{Z}/223))[t]/(t^{37} - 2)$.
$(\mathbf{Z}/2)[t]/(t^{283} - t^{12} - t^7 - t^5 - 1)$.

How quickly can we
add, subtract, multiply
in these fields?

Answer will depend on platform:
AMD Athlon, Sun UltraSPARC IV,
Intel 8051, Xilinx Spartan-3, etc.
Warning: different platforms
often favor different fields!

The first question

How to multiply big integers?

Child's answer: Use polynomial with coefficients in $\{0, 1, \ldots, 9\}$ to represent integer in radix 10.

With this representation, multiply integers in two steps:
1. Multiply polynomials.
2. "Carry" extra digits.

Polynomial multiplication involves *small* integers.
Have split one big multiplication into many small operations.

Example of representation:

$839 = 8 \cdot 10^2 + 3 \cdot 10^1 + 9 \cdot 10^0 =$ value (at $t = 10$) of polynomial $8t^2 + 3t^1 + 9t^0$.

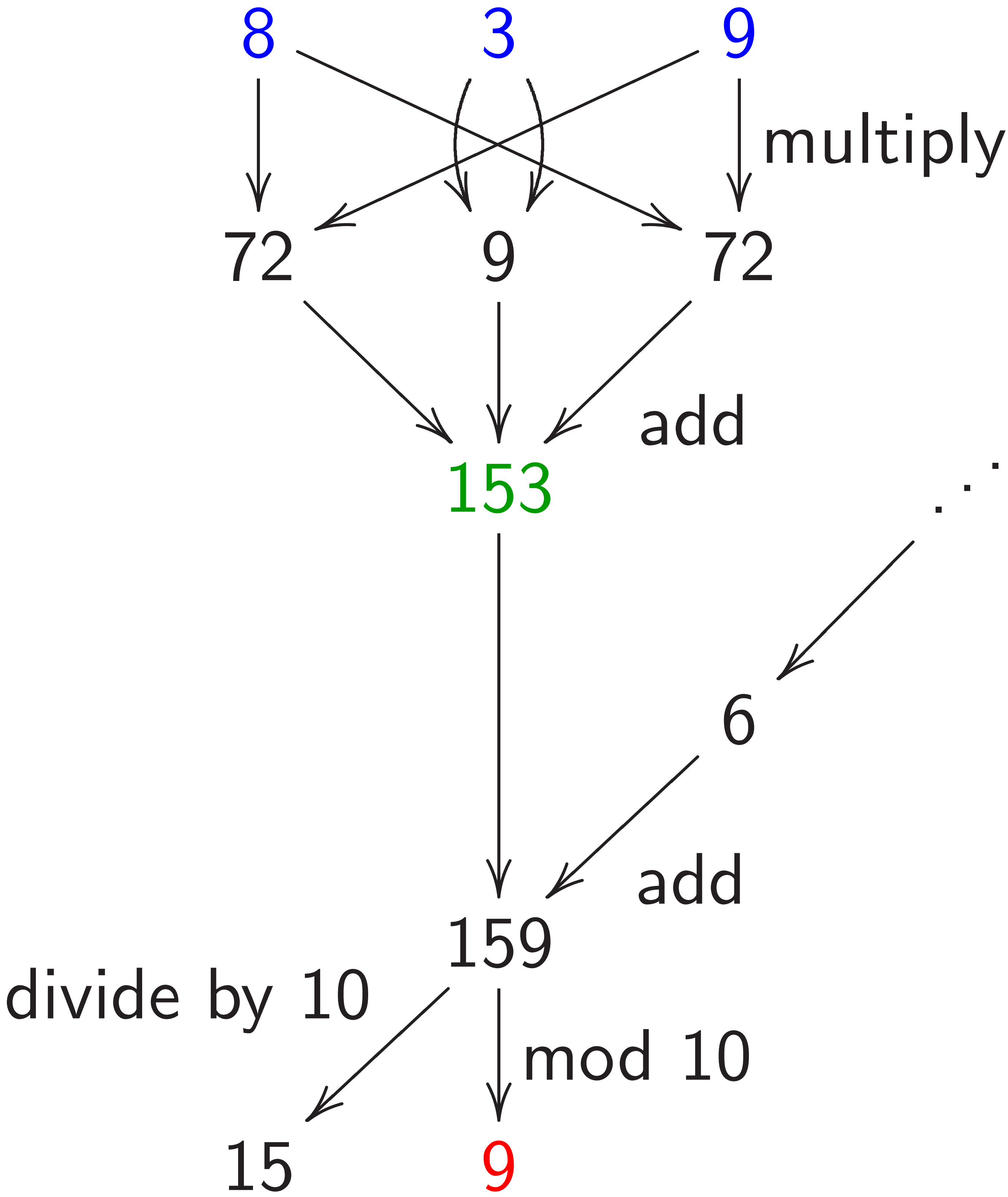
Squaring: $(8t^2 + 3t^1 + 9t^0)^2 =$
$64t^4 + 48t^3 + 153t^2 + 54t^1 + 81t^0$.

Carrying:

$64t^4 + 48t^3 + 153t^2 + 54t^1 + 81t^0$;
$64t^4 + 48t^3 + 153t^2 + 62t^1 + 1t^0$;
$64t^4 + 48t^3 + 159t^2 + 2t^1 + 1t^0$;
$64t^4 + 63t^3 + 9t^2 + 2t^1 + 1t^0$;
$70t^4 + 3t^3 + 9t^2 + 2t^1 + 1t^0$;
$7t^5 + 0t^4 + 3t^3 + 9t^2 + 2t^1 + 1t^0$.

In other words, $839^2 = 703921$.

What operations were used here?



Scaled variation:

$839 = 800 + 30 + 9 =$

value (at $t = 1$) of polynomial

$800t^2 + 30t^1 + 9t^0$.

Squaring: $(800t^2 + 30t^1 + 9t^0)^2 =$

$640000t^4 + 48000t^3 + 15300t^2 +$

$540t^1 + 81t^0$.

Carrying:

$640000t^4 + 48000t^3 + 15300t^2 +$

$540t^1 + 81t^0$;

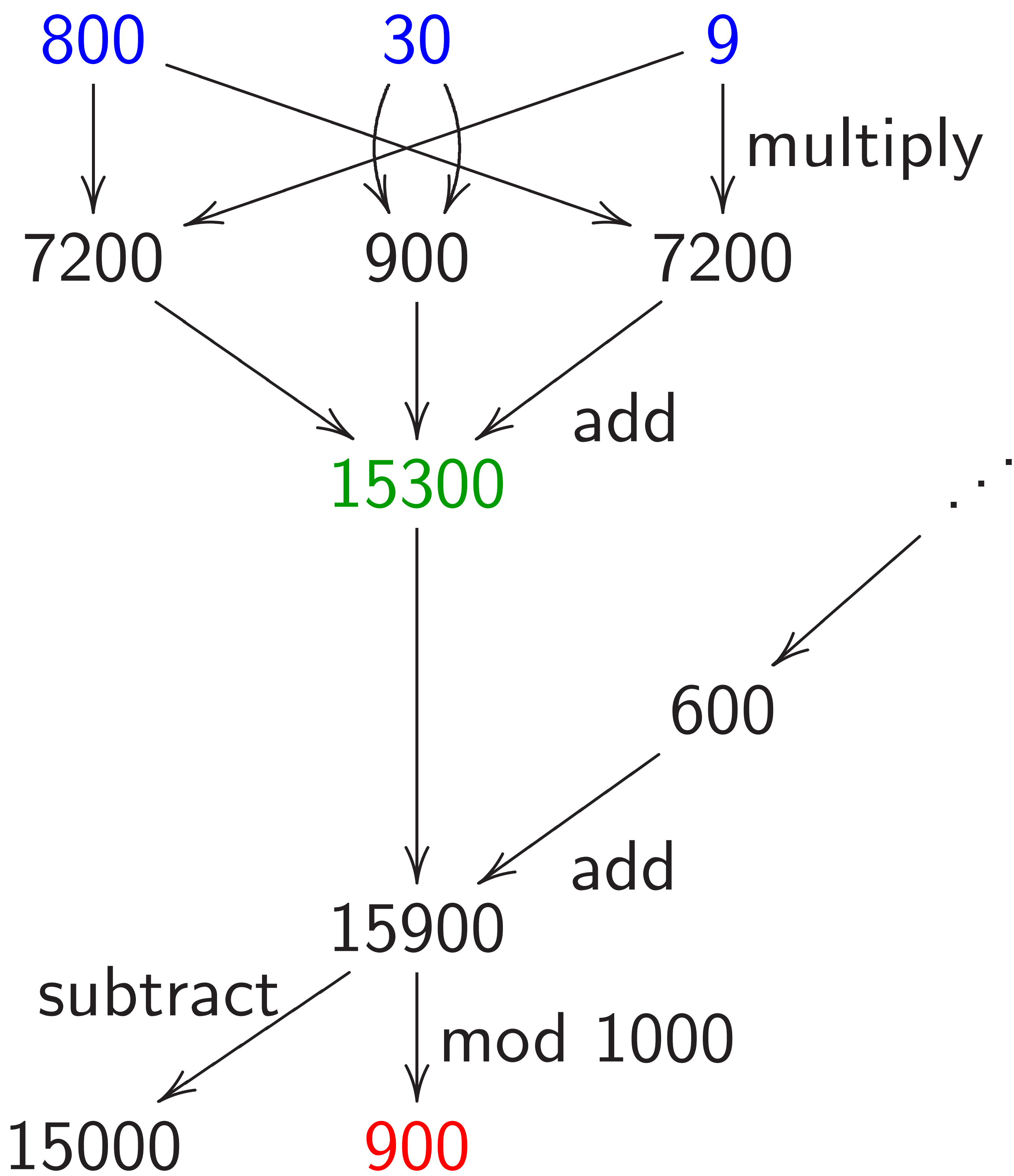
$640000t^4 + 48000t^3 + 15300t^2 +$

$620t^1 + 1t^0$; $\ldots$

$700000t^5 + 0t^4 + 3000t^3 + 900t^2 +$

$20t^1 + 1t^0$.

What operations were used here?



Speedup: double inside squaring

Squaring $\cdots + f_2 t^2 + f_1 t^1 + f_0 t^0$ produces coefficients such as $f_4 f_0 + f_3 f_1 + f_2 f_2 + f_1 f_3 + f_0 f_4$.

Compute more efficiently as $2 f_4 f_0 + 2 f_3 f_1 + f_2 f_2$.
Or, slightly faster, $2(f_4 f_0 + f_3 f_1) + f_2 f_2$.
Or, slightly faster, $(2 f_4) f_0 + (2 f_3) f_1 + f_2 f_2$ after precomputing $2 f_1, 2 f_2, \ldots$.

Have eliminated $\approx 1/2$ of the work if there are many coefficients.

Speedup: allow negative coeffs

Recall $159 \mapsto 15, 9$.

Scaled: $15900 \mapsto 15000, 900$.

Alternative: $159 \mapsto 16, -1$.

Scaled: $15900 \mapsto 16000, -100$.

Use digits $\{-5, -4, \ldots, 4, 5\}$ instead of $\{0, 1, \ldots, 9\}$.

Several small advantages:
easily handle negative integers;
easily handle subtraction;
reduce products a bit.

Speedup: delay carries

Computing (e.g.) big $ab + c^2$: multiply a, b polynomials, carry, square c poly, carry, add, carry.

e.g. $a = 314$, $b = 271$, $c = 839$:
$(3t^2 + 1t^1 + 4t^0)(2t^2 + 7t^1 + 1t^0) =$
$6t^4 + 23t^3 + 18t^2 + 29t^1 + 4t^0$;
carry: $8t^4 + 5t^3 + 0t^2 + 9t^1 + 4t^0$.

As before $(8t^2 + 3t^1 + 9t^0)^2 =$
$64t^4 + 48t^3 + 153t^2 + 54t^1 + 81t^0$;
$7t^5 + 0t^4 + 3t^3 + 9t^2 + 2t^1 + 1t^0$.

$+$: $7t^5 + 8t^4 + 8t^3 + 9t^2 + 11t^1 + 5t^0$;
$7t^5 + 8t^4 + 9t^3 + 0t^2 + 1t^1 + 5t^0$.

Faster: multiply a, b polynomials, square c polynomial, add, carry.

$(6t^4 + 23t^3 + 18t^2 + 29t^1 + 4t^0) +$
$(64t^4 + 48t^3 + 153t^2 + 54t^1 + 81t^0) =$
$70t^4 + 71t^3 + 171t^2 + 83t^1 + 85t^0;$
$7t^5 + 8t^4 + 9t^3 + 0t^2 + 1t^1 + 5t^0.$

Eliminate intermediate carries. Outweighs cost of handling slightly larger coefficients.

Important to carry between multiplications (and squarings) to reduce coefficient size; but carries are usually a bad idea for additions, subtractions, etc.

Speedup: polynomial Karatsuba

Computing product of polys f, g with (e.g.) $\deg f < 20$, $\deg g < 20$: 400 coefficient mults, 361 coefficient adds.

Faster: Write f as $F_0 + F_1 t^{10}$ with $\deg F_0 < 10$, $\deg F_1 < 10$. Similarly write g as $G_0 + G_1 t^{10}$.

Then $fg = (F_0 + F_1)(G_0 + G_1)t^{10}$ $+ (F_0 G_0 - F_1 G_1 t^{10})(1 - t^{10})$.

20 adds for $F_0 + F_1$, $G_0 + G_1$.
300 mults for three products
$F_0 G_0$, $F_1 G_1$, $(F_0 + F_1)(G_0 + G_1)$.
243 adds for those products.
9 adds for $F_0 G_0 - F_1 G_1 t^{10}$
with subs counted as adds
and with delayed negations.
19 adds for $\cdots (1 - t^{10})$.
19 adds to finish.

Total 300 mults, 310 adds.
Larger coefficients, slight expense;
still saves time.

Can apply idea recursively
as poly degree grows.

Many other algebraic speedups
in polynomial multiplication:
Toom, FFT, etc.

Increasingly important as
polynomial degree grows.
$O(n \lg n \lg \lg n)$ coeff operations
to compute n-coeff product.

Useful for sizes of n
that occur in cryptography?
Maybe; active research area.

Using CPU's integer instructions

Replace radix 10 with, e.g., 2^{24}.
Power of 2 simplifies carries.

Adapt radix to platform.

e.g. Every 2 cycles, Athlon 64 can compute a 128-bit product of two 64-bit integers.
(5-cycle latency; parallelize!)
Also low cost for 128-bit add.

Reasonable to use radix 2^{60}.
Sum of many products of digits fits comfortably below 2^{128}.
Be careful: analyze largest sum.

e.g. In 4 cycles, Intel 8051
can compute a 16-bit product
of two 8-bit integers.
Could use radix 2^6.
Could use radix 2^8,
with 24-bit sums.

e.g. Every 2 cycles, Pentium 4 F3
can compute a 64-bit product
of two 32-bit integers.
(11-cycle latency; yikes!)
Reasonable to use radix 2^{28}.

Warning: Multiply instructions
are very slow on some CPUs.
e.g. Pentium 4 F2: 10 cycles!

Using floating-point instructions

Big CPUs have separate floating-point instructions, aimed at numerical simulation but useful for cryptography.

In my experience, floating-point instructions support faster multiplication (often much, much faster) than integer instructions, except on the Athlon 64. Other advantages: portability; easily scaled coefficients.

e.g. Every 2 cycles, Pentium III can compute a 64-bit product of two floating-point numbers, and an independent 64-bit sum.

e.g. Every cycle, Athlon can compute a 64-bit product and an independent 64-bit sum.

e.g. Every cycle, UltraSPARC III can compute a 53-bit product and an independent 53-bit sum. Reasonable to use radix 2^{24}.

e.g. Pentium 4 can do the same using SSE2 instructions.

How to do carries in floating-point registers? (No CPU carry instruction: not useful for simulations.)

Exploit floating-point rounding: add big constant, subtract same constant.

e.g. Given α with $|\alpha| \le 2^{75}$: compute 53-bit floating-point sum of α and constant $3 \cdot 2^{75}$, obtaining a multiple of 2^{24}; subtract $3 \cdot 2^{75}$ from result, obtaining multiple of 2^{24} nearest α; subtract from α.

Reducing modulo a prime

Fix a prime p.
The prime field $\mathbf{Z}/p$
is the set $\{0, 1, 2, \ldots, p-1\}$
with $-$ defined as $-$ mod p,
$+$ defined as $+$ mod p,
$\cdot$ defined as $\cdot$ mod p.

e.g. $p = 1000003$:
$1000000 + 50 = 47$ in $\mathbf{Z}/p$;
$-1 = 1000002$ in $\mathbf{Z}/p$;
$117505 \cdot 23131 = 1$ in $\mathbf{Z}/p$.

How to multiply in $\mathbf{Z}/p$?

Can use definition:
$fg \bmod p = fg - p\lfloor fg/p\rfloor$.
Can multiply fg by a precomputed $1/p$ approximation; easily adjust to obtain $\lfloor fg/p\rfloor$.
Slight speedup: "2-adic inverse"; "Montgomery reduction."

We can do better: normally p is chosen with a special form (or dividing a special form; see "redundant representations") to make $fg \bmod p$ much faster.

e.g. In $\mathbf{Z}/1000003$:

$314159265358 =$

$314159 \cdot 1000000 + 265358 =$

$314159(-3) + 265358 =$

$-942477 + 265358 =$

$-677119.$

Easily adjust to range

$\{0, 1, \ldots, p - 1\}$

by adding/subtracting a few p's.

(Beware timing attacks!)

Speedup: Delay the adjustment;

extra p's won't damage

subsequent field operations.

Can delay carries until after multiplication by 3.

e.g. To square 314159 in $\mathbf{Z}/1000003$: Square poly $3t^5 + 1t^4 + 4t^3 + 1t^2 + 5t^1 + 9t^0$, obtaining $9t^{10} + 6t^9 + 25t^8 + 14t^7 + 48t^6 + 72t^5 + 59t^4 + 82t^3 + 43t^2 + 90t^1 + 81t^0$.

Reduce: replace $(c_i)t^{6+i}$ by $(-3c_i)t^i$, obtaining $72t^5 + 32t^4 + 64t^3 - 32t^2 + 48t^1 - 63t^0$.

Carry: $8t^6 - 4t^5 - 2t^4 + 1t^3 + 2t^2 + 2t^1 - 3t^0$.

To minimize poly degree, mix reduction and carrying, carrying the top sooner.

e.g. Start from square $9t^{10} + 6t^9 + 25t^8 + 14t^7 + 48t^6 + 72t^5 + 59t^4 + 82t^3 + 43t^2 + 90t^1 + 81t^0$.

Reduce $t^{10} \to t^4$ and carry $t^4 \to t^5 \to t^6$: $6t^9 + 25t^8 + 14t^7 + 56t^6 - 5t^5 + 2t^4 + 82t^3 + 43t^2 + 90t^1 + 81t^0$.

Finish reduction: $-5t^5 + 2t^4 + 64t^3 - 32t^2 + 48t^1 - 87t^0$. Carry $t^0 \to t^1 \to t^2 \to t^3 \to t^4 \to t^5$: $-4t^5 - 2t^4 + 1t^3 + 2t^2 - 1t^1 + 3t^0$.

Speedup: non-integer radix

Consider $\mathbf{Z}/(2^{61}-1)$.

Five coeffs in radix 2^{13}?
$f_4t^4 + f_3t^3 + f_2t^2 + f_1t^1 + f_0t^0$.
Most coeffs could be 2^{12}.

Square $\cdots + 2(f_4f_1 + f_3f_2)t^5 + \cdots$.
Coeff of t^5 could be $> 2^{25}$.

Reduce: $2^{65} = 2^4$ in $\mathbf{Z}/(2^{61}-1)$;
$\cdots + (2^5(f_4f_1 + f_3f_2) + f_0^2)t^0$.
Coeff could be $> 2^{29}$.
Very little room for
additions, delayed carries, etc.
on 32-bit platforms.

Scaled: Evaluate at $t = 1$.

f_4 is multiple of 2^{52};

f_3 is multiple of 2^{39};

f_2 is multiple of 2^{26};

f_1 is multiple of 2^{13};

f_0 is multiple of 2^{0}. Reduce:

$\cdots + (2^{-60}(f_4 f_1 + f_3 f_2) + f_0^2)t^0$.

Better: Non-integer radix $2^{12.2}$.

f_4 is multiple of 2^{49};

f_3 is multiple of 2^{37};

f_2 is multiple of 2^{25};

f_1 is multiple of 2^{13};

f_0 is multiple of 2^{0}.

Saves a few bits in coeffs.

More finite fields

Fix a prime p. Fix a poly φ in one variable t with φ irreducible mod p.

The finite field $(\mathbf{Z}/p)[t]/\varphi$ is the set of polynomials $f_{\deg\varphi-1}t^{\deg\varphi-1}+\cdots+f_1t^1+f_0t^0$ with each $f_i\in\mathbf{Z}/p$ and with $-,+,\cdot$ defined modulo p and modulo φ.

$(\mathbf{Z}/p)[t]/\varphi$ is an "extension" of the prime field $\mathbf{Z}/p$; it has "characteristic" p.

e.g. 223 is prime, and poly
$t^6 - 3$ is irreducible mod 223,
so $(\mathbf{Z}/223)[t]/(t^6 - 3)$ is a field.

223^6 elements of field,
namely polynomials $f_5 t^5 + f_4 t^4 + f_3 t^3 + f_2 t^2 + f_1 t^1 + f_0 t^0$
with each $f_i \in \{0, 1, \ldots, 222\}$.

After adding, subtracting,
multiplying: replace t^6 by 3,
replace t^7 by $3t$, etc.; and
reduce coefficients modulo 223.
e.g. $(9t^4 + 1)^2 = 81t^8 + 18t^4 + 1 = 243t^2 + 18t^4 + 1 = 18t^4 + 20t^2 + 1$.

Have two levels of polynomials when p is large: element of $(\mathbf{Z}/p)[t]/\varphi$ is poly mod φ; each poly coefficient is integer represented as poly in some radix.

e.g. $f_4 t^4 + f_3 t^3 + f_2 t^2 + f_1 t^1 + f_0 t^0$ in $(\mathbf{Z}/(2^{61}-1))[t]/(t^5-3)$ could have each coefficient f_i represented as poly of degree < 3 in radix $2^{61/3}$.

When p is small, especially $p = 2$, benefit from batching coefficients. Many platform-specific speedups.

Speedup: fast Frobenius

In $(\mathbf{Z}/2)[t]/\varphi$ have
$(\cdots + f_2 t^2 + f_1 t^1 + f_0 t^0)^2 =$
$\cdots + f_2 t^4 + f_1 t^2 + f_0 t^0$.

Cross-terms disappear: $2 = 0$.

Thus squaring is very fast:
replace t^i by t^{2i} and
reduce modulo φ.

More generally, pth powering
is very fast in characteristic p.