REDUCING LATTICE BASES TO FIND SMALL-HEIGHT VALUES OF UNIVARIATE POLYNOMIALS

DANIEL J. BERNSTEIN

ABSTRACT. This paper generalizes several previous results on finding divisors in residue classes (Lenstra, Konyagin, Pomerance, Coppersmith, Howgrave-Graham, Nagaraj), finding divisors in intervals (Rivest, Shamir, Coppersmith, Howgrave-Graham), finding modular roots (Hastad, Vallée, Girault, Toffin, Coppersmith, Howgrave-Graham), finding high-power divisors (Boneh, Durfee, Howgrave-Graham), and finding codeword errors beyond half distance (Sudan, Guruswami, Goldreich, Ron, Boneh) into a unified algorithm that, given f and g, finds all rational numbers r such that $f(r)$ and $g(r)$ both have small height.

1. Introduction

Consider the fraction $(r^3-s)/n$, where n is a large integer with no known factors. Usually there is no cancellation between the numerator r^3-s and the denominator n. In other words, the height of $(r^3-s)/n$ is usually $\max\{\left|r^3-s\right|, n\}$. Here the **height** of a rational number m/n is, by definition, $\max\{|m|, |n|\}/\gcd\{m, n\}$.

However, if r is a cube root of s modulo n, then one can remove n from both the numerator and denominator. In other words, the height of $(r^3-s)/n$ is only $\max\{\left|(r^3-s)/n\right|, 1\}$. The problem of finding a cube root of s modulo n can thus be viewed as the problem of finding small-height values of the polynomial $(x^3-s)/n$.

Many other useful properties of numbers r can be recast in the form "$f(r)$ has small height" for various polynomials f. For example, the problem of factoring n can be viewed as the problem of finding all r such that r/n has small height.

There is a surprisingly fast method, using lattice-basis reduction, to find all numbers r such that *both* r and $f(r)$ have small height. This paper presents a very general statement of the method (see Theorem 2.3); asymptotically optimal parameters (see Section 3); and an exposition of various applications of the method (see Sections 4, 5, and 7). The theorems and algorithms can easily be switched from $\mathbf{Q}$ to the rational function field $\mathbf{F}_q(t)$ over a finite field $\mathbf{F}_q$, although better algorithms are often available in the function-field case.

I have made no attempt to cover analogous methods for higher-degree global fields or for polynomials in more variables. There are several papers on small-height values of bivariate polynomials, but each application seems to pose a new optimization problem. I will leave it to future writers to unify the literature on this topic.

Date: 2006.05.31. Permanent ID of this document: `82f82c041b7e2bdce94a5e1f94511773`.

2000 *Mathematics Subject Classification.* Primary 11Y16. Secondary 94B35.

The author was supported by the National Science Foundation under grant DMS–0140542, and by the Alfred P. Sloan Foundation.

History. The following table fits previous results into the framework of Theorem 2.3. Notation: f is the polynomial with useful small-height values; d is the degree of f; m is the lattice rank; k is the highest f exponent used in defining the lattice. Results improve primarily as m increases, secondarily as k increases.

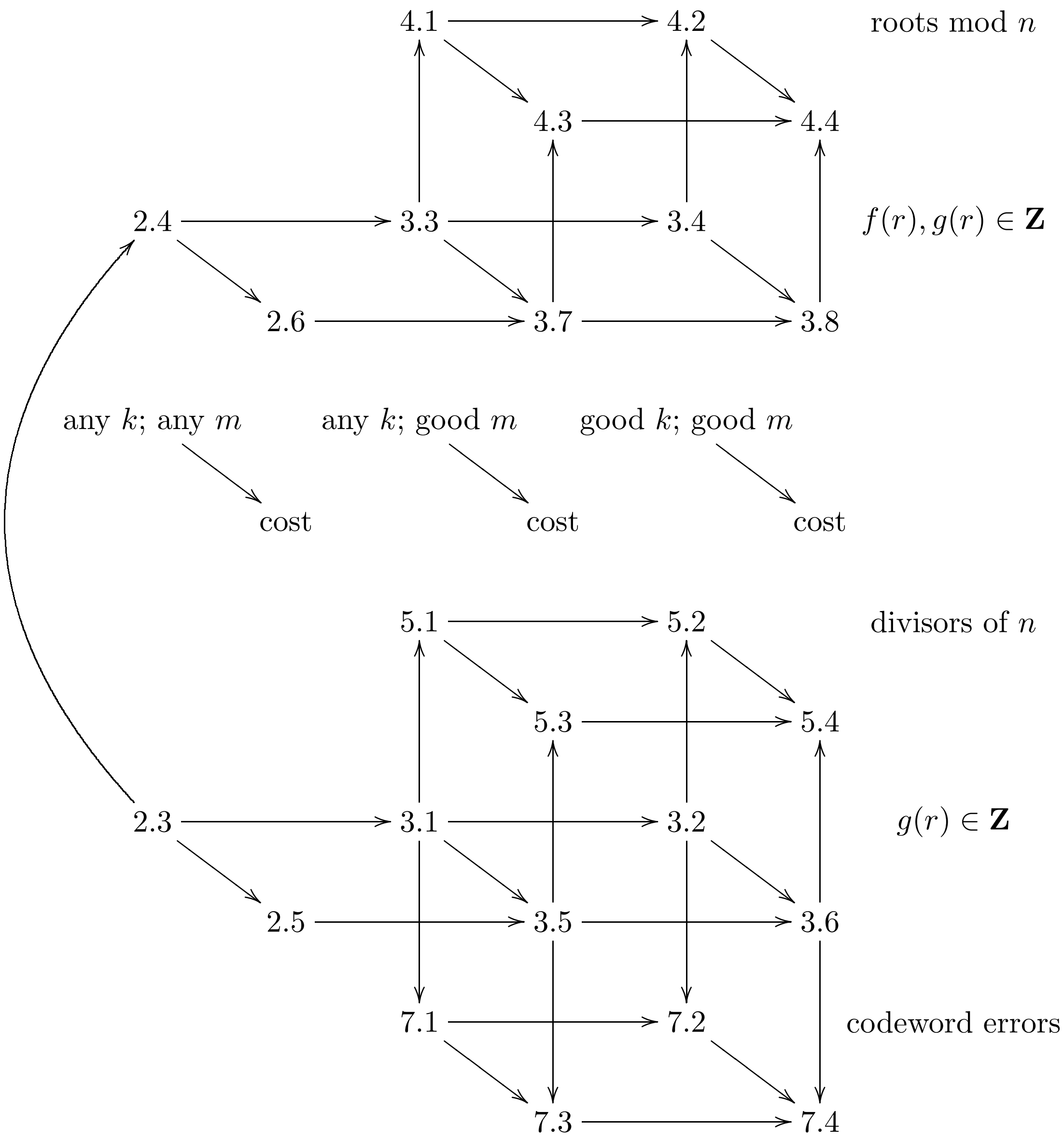
Find	$f(r)$	k	m	Notes
divisors, in $u+v\mathbf{Z}$, of n	$(r+uw)/n$ where $wv \in 1+n\mathbf{Z}$	1	3	1984 Lenstra [24], for proving primality
divisors, in an interval, of n	$(r+w)/n$ for one w	1	3	1986 Rivest Shamir [31], for breaking cryptosystems; independent of [24]
roots of $p(x)$ mod n	$p(r)/n$	1	$d+1$	1988 Håstad [18, Section 3]; first use of nonlinear f; independently: 1989 Vallée Girault Toffin [34] (using the dual lattice; more difficult)
roots of $p(x)$ mod n	$p(r)/n$	big	big	1996 Coppersmith [8] (using dual), for breaking cryptosystems; first use of increasing m; first use of increasing k; simplified: 1997 Howgrave-Graham [19] (explicitly avoiding dual)
divisors, in an interval, of n	$(r+w)/n$	big	big	1996 Coppersmith [9] (in a much more complicated way); simplified: 1997 Howgrave-Graham [19]
divisors, in $1+v\mathbf{Z}$, of n	$(r+w)/n$	2	5	1997 Konyagin Pomerance [22, Algorithm 3.2]; independent of [8]
divisors, in $u+v\mathbf{Z}$, of n	$(r+uw)/n$	big	big	1998 Coppersmith Howgrave-Graham Nagaraj [20, Section 5.5]
large values of $\gcd\{x+w,n\}$	$(r+w)/n$	1	big	1999 Goldreich Ron Sudan [14] (using dual), for error correction; previous function-field version: 1997 Sudan [33]; independent of [8]
high-power divisors, in an interval, of n	$(r+w)^d/n$	big	big	1999 Boneh Durfee Howgrave-Graham [7]
large values of $\gcd\{x+w,n\}$	$(r+w)/n$	big	big	2000 Boneh [5], for error correction; independently: 2001 Howgrave-Graham [21, Section 3]; previous function-field version: 1999 Guruswami Sudan [17]
large values of $\gcd\{p(x),n\}$	$p(r)/n$	big	big	2000 Boneh [5, Section 4]

It was recognized in [19] and [7] that "$r+w$ divides n" and "$(r+w)^d$ divides n" could be handled by the same technique as "$p(r)$ is divisible by n." Meanwhile, "$\gcd\{r+w,n\}$ is large" appeared independently in [14]. A unified "$\gcd\{p(r),n\}$ is large" algorithm finally appeared, with insufficient fanfare, in [5, Section 4].

Index of theorems in this paper. Algorithms in this paper are expressed in two ways: as theorems stating that the algorithms produce the desired results, and as "cost" theorems stating that there exist low-cost algorithms (in a particular cost measure) producing the desired results. Readers who want to understand what the algorithms achieve, without worrying at first about how the algorithms work, should start with the cost theorems, such as Theorem 4.4.

The following chart has three rows for algorithms aimed at specific applications: "roots mod n," "divisors of n," and "codeword errors." It also has two rows for more general algorithms that can be used for other applications: an "$f(r), g(r) \in \mathbf{Z}$" row generalizing "roots mod n," and a "$g(r) \in \mathbf{Z}$" row generalizing all of these applications.

Algorithms in the "any k; any m" column have two parameters (k, m) affecting their speed and output; the user can tune these parameters for the application at hand. Algorithms in the "good k; good m" column fix choices of (k, m) that work reasonably well for a wide variety of applications, although they are often not exactly optimal. Readers who find themselves overwhelmed by the flexibility of k and m should start with the algorithms in the "good k; good m" column.



2. The general method

This section explains how to find all rational numbers r such that $f(r)$ and $g(r)$ simultaneously have small height. Here $f, g \in \mathbf{Q}[x]$ are polynomials, each of positive degree, each with positive leading coefficient. Write $d = \deg f$, and assume for simplicity that $\deg g = 1$.

Theorem 2.2 below gives a more precise definition of "small height." The height bound depends on two integer parameters $k \ge 1$ and $m \ge dk+1$. A typical special case is $k = 1$ and $m = 2d$. See Section 3 for further comments on the choice of k and m.

The lattice. Define $L \subset \mathbf{Q}[x]$ as the $\mathbf{Z}$-module

$$\begin{array}{llllllllll} & \mathbf{Z} & + & \mathbf{Z}g & + & \mathbf{Z}g^2 & + & \cdots & + & \mathbf{Z}g^{d-1} \\ + & \mathbf{Z}f & + & \mathbf{Z}gf & + & \mathbf{Z}g^2 f & + & \cdots & + & \mathbf{Z}g^{d-1} f \\ + & \mathbf{Z}f^2 & + & \mathbf{Z}gf^2 & + & \mathbf{Z}g^2 f^2 & + & \cdots & + & \mathbf{Z}g^{d-1} f^2 \\ + & \cdots & & & & & & & & \\ + & \mathbf{Z}f^{k-1} & + & \mathbf{Z}gf^{k-1} & + & \mathbf{Z}g^2 f^{k-1} & + & \cdots & + & \mathbf{Z}g^{d-1} f^{k-1} \\ + & \mathbf{Z}f^k & + & \cdots & + & \mathbf{Z}g^{m-dk-1} f^k. & & & & \end{array}$$

For example, if $k = 1$ and $m = d+1$, then $L = \mathbf{Z} + \mathbf{Z}g + \mathbf{Z}g^2 + \cdots + \mathbf{Z}g^{d-1} + \mathbf{Z}f$; if $k = 1$ and $m = 2d$, then $L = \mathbf{Z} + \mathbf{Z}g + \mathbf{Z}g^2 + \cdots + \mathbf{Z}g^{d-1} + \mathbf{Z}f + \cdots + \mathbf{Z}g^{d-1}f$.

The basis elements $1, g, \ldots, g^{d-1}, f, \ldots, g^{m-dk-1} f^k$ have degrees $0, 1, 2, \ldots, m-1$ respectively. Thus L is a lattice of rank m under the usual coefficient-vector metric on $\mathbf{Q}[x]$, namely $\varphi \mapsto |\varphi| = \sqrt{\varphi_0^2 + \varphi_1^2 + \varphi_2^2 + \cdots}$, where $\varphi = \varphi_0 + \varphi_1 x + \varphi_2 x^2 + \cdots$.

The basis elements have leading coefficients $1, g_1, g_1^2, \ldots, g_1^{m-dk-1} f_d^k$, where g_1 is the leading coefficient of g and f_d is the leading coefficient of f. Thus

$$\begin{aligned} \det L &= g_1^{kd(d-1)/2 + (m-dk)(m-dk-1)/2} f_d^{dk(k-1)/2 + k(m-dk)} \\ &= g_1^{m(m-1)/2} (g_1^d / f_d)^{dk(k+1)/2 - mk}. \end{aligned}$$

For example, if $k = 1$ and $m = 2d$, then $\det L = g_1^{d(d-1)} f_d^d = g_1^{d(2d-1)} (g_1^d/f_d)^{-d}$.

Theorem 2.1. *Let d, k, m be positive integers with $m \ge dk+1$. Let $f \in \mathbf{Q}[x]$ be a polynomial of degree d with leading coefficient $f_d > 0$. Let $g \in \mathbf{Q}[x]$ be a polynomial of degree 1 with leading coefficient $g_1 > 0$. Define L as above. If $\varphi \in L$, $r \in \mathbf{Q}$, and $\gcd\{1, f(r)\}^k \gcd\{1, g(r)\}^{\max\{d-1, m-dk-1\}} > \left|(1, r, \ldots, r^{m-1})\right| |\varphi|$, then $\varphi(r) = 0$.*

For example, if $k = 1$, $m = 2d$, $\varphi \in L$, $r \in \mathbf{Q}$, and $\gcd\{1, f(r)\} \gcd\{1, g(r)\}^{d-1} > \left|(1, r, \ldots, r^{2d-1})\right| |\varphi|$, then $\varphi(r) = 0$.

The reader should interpret $\gcd\{1, f(r)\} > \cdots$ as "$f(r)$ has small denominator"; $\gcd\{1, g(r)\} > \cdots$ as "$g(r)$ has small denominator"; and $\left|(1, r, \ldots, r^{m-1})\right| < \cdots$ as "$f(r)$ and $g(r)$ have small numerators." Theorem 2.1 can thus be summarized as "$\varphi(r) = 0$ if $f(r)$ and $g(r)$ both have small height."

Proof. $|\varphi(r)| \le \left|(1, r, \ldots, r^{m-1})\right| |\varphi| < \gcd\{1, f(r)\}^k \gcd\{1, g(r)\}^{\max\{d-1, m-dk-1\}}$. But $\varphi \in \mathbf{Z} + \mathbf{Z}g + \cdots + \mathbf{Z}g^{d-1} f^{k-1} + \mathbf{Z}f^k + \cdots + \mathbf{Z}g^{m-dk-1} f^k$ by definition of L, so $\varphi(r) \in \mathbf{Z} + \mathbf{Z}g(r) + \cdots + \mathbf{Z}g(r)^{d-1} f(r)^{k-1} + \mathbf{Z}f(r)^k + \cdots + \mathbf{Z}g(r)^{m-dk-1} f(r)^k \subseteq \mathbf{Z} \gcd\{1, f(r)\}^k \gcd\{1, g(r)\}^{\max\{d-1, m-dk-1\}}$. Thus $\varphi(r) = 0$. $\square$

Theorem 2.2. *Let d, k, m be positive integers with $m \ge dk+1$. Let $f \in \mathbf{Q}[x]$ be a polynomial of degree d with leading coefficient $f_d > 0$. Let $g \in \mathbf{Q}[x]$ be a polynomial of degree 1 with leading coefficient $g_1 > 0$. Define L as above. Let $\varphi \in L$ be a nonzero vector such that $|\varphi| \le 2^{(m-1)/2}(\det L)^{1/m}$. If $r \in \mathbf{Q}$ and*

$$\gcd\{1, f(r)\}^k \gcd\{1, g(r)\}^{\max\{d-1, m-dk-1\}} > \left|(1, r, \ldots, r^{m-1})\right| (2g_1)^{(m-1)/2}(g_1^d/f_d)^{dk(k+1)/2m-k}$$

then $\varphi(r) = 0$.

Proof. $(\det L)^{1/m} = g_1^{(m-1)/2}(g_1^d/f_d)^{dk(k+1)/2m-k}$. Apply Theorem 2.1. □

For example, if $k = 1$ and $m = 2d$, then $\varphi(r) = 0$ for every $r \in \mathbf{Q}$ such that $\gcd\{1, f(r)\}\gcd\{1, g(r)\}^{d-1} > \left|(1, r, \ldots, r^{2d-1})\right| (2g_1)^{d-1/2}(g_1^d/f_d)^{-1/2}$.

Theorem 2.3. *Let d, k, m be positive integers with $m \ge dk+1$. Let $f \in \mathbf{Q}[x]$ be a polynomial of degree d with leading coefficient $f_d > 0$. Let $g \in \mathbf{Q}[x]$ be a polynomial of degree 1 with leading coefficient $g_1 > 0$. Define L as above. Let $\varphi \in L$ be a nonzero vector such that $|\varphi| \le 2^{(m-1)/2}(\det L)^{1/m}$. Define $\gamma = m^{1/2k}(2g_1)^{(m-1)/2k}(g_1^d/f_d)^{d(k+1)/2m-1}$. If $r \in \mathbf{Q}$, $|r| \le 1$, $\gcd\{1, f(r)\} > \gamma$, and $g(r) \in \mathbf{Z}$, then $\varphi(r) = 0$.*

For example, if $k = 1$ and $m = 2d$, then $\varphi(r) = 0$ for every $r \in \mathbf{Q}$ such that $|r| \le 1$, $g(r) \in \mathbf{Z}$, and $\gcd\{1, f(r)\} > \gamma$, where $\gamma = (2d)^{1/2}(2g_1)^{d-1/2}(g_1^d/f_d)^{-1/2}$.

Proof. $\gamma^k = m^{1/2}(2g_1)^{(m-1)/2}(g_1^d/f_d)^{dk(k+1)/2m-k}$; $\left|(1, r, \ldots, r^{m-1})\right| \le m^{1/2}$; and $\gcd\{1, g(r)\} = 1$. Apply Theorem 2.2. □

Theorem 2.4. *Let d, k, m be positive integers with $m \ge dk+1$. Let $f \in \mathbf{Q}[x]$ be a polynomial of degree d with leading coefficient $f_d > 0$. Let $g \in \mathbf{Q}[x]$ be a polynomial of degree 1 with leading coefficient $g_1 > 0$. Define L as above. Let $\varphi \in L$ be a nonzero vector such that $|\varphi| \le 2^{(m-1)/2}(\det L)^{1/m}$. Assume that $g_1 < (g_1^d/f_d)^{2k/(m-1)-dk(k+1)/m(m-1)}/2m^{1/(m-1)}$. If $r \in \mathbf{Q}$, $|r| \le 1$, $f(r) \in \mathbf{Z}$, and $g(r) \in \mathbf{Z}$, then $\varphi(r) = 0$.*

For example, if $k = 1$ and $m = 2d$, then $\varphi(r) = 0$ for every $r \in \mathbf{Q}$ such that $|r| \le 1$, $f(r) \in \mathbf{Z}$, and $g(r) \in \mathbf{Z}$, provided that $2g_1 < (g_1^d/2df_d)^{1/(2d-1)}$.

Proof. By assumption $m^{1/2k}(2g_1)^{(m-1)/2k}(g_1^d/f_d)^{d(k+1)/2m-1} < 1 = \gcd\{1, f(r)\}$. Apply Theorem 2.3. □

Computation. It is easy to compute the rational numbers r identified in Theorems 2.2, 2.3, and 2.4:

- Feed the basis vectors $1, g, \ldots, g^{d-1}, f, \ldots, g^{m-dk-1}f^k$ of L to a lattice-basis-reduction algorithm, such as the Lenstra-Lenstra-Lovasz algorithm, to obtain a nonzero vector $\varphi \in L$ such that $|\varphi| \le 2^{(m-1)/2}(\det L)^{1/m}$. See [23] or [25]. The theorems now state that all desired numbers r are roots of φ.
- Compute the rational roots of φ, by approximating the real (or 2-adic) roots of φ to high precision. See, e.g., [26]. By construction φ has degree at most $m-1$, so it has at most $m-1$ roots.
- Check each root r to see whether it satisfies the stated conditions.

Each step is reasonably fast if f, g, k, and m are reasonably small.

One way to measure the complexity of this algorithm is to measure its output size, i.e., to count the number of qualifying r's. Theorems 2.5 and 2.6 state bounds on this measure of algorithm complexity. I will leave it to the reader to formulate theorems regarding other measures.

Theorem 2.5. *Let d, k, m be positive integers with $m \ge dk + 1$. Let $f \in \mathbf{Q}[x]$ be a polynomial of degree d with leading coefficient $f_d > 0$. Let $g \in \mathbf{Q}[x]$ be a polynomial of degree 1 with leading coefficient $g_1 > 0$. Then there are at most $m - 1$ values $r \in \mathbf{Q}$ such that $|r| \le 1$, $\gcd\{1, f(r)\} > m^{1/2k}(2g_1)^{(m-1)/2k}(g_1^d/f_d)^{d(k+1)/2m-1}$, and $g(r) \in \mathbf{Z}$.*

Take, for example, $k = 1$ and $m = 2d$: there are at most $2d - 1$ values $r \in \mathbf{Q}$ such that $|r| \le 1$, $\gcd\{1, f(r)\} > (2d)^{1/2}(2g_1)^{d-1/2}(g_1^d/f_d)^{-1/2}$, and $g(r) \in \mathbf{Z}$.

Proof. Apply lattice-basis reduction to Theorem 2.3.

In more detail: Define $\gamma = m^{1/2k}(2g_1)^{(m-1)/2k}(g_1^d/f_d)^{d(k+1)/2m-1}$, and define L as above. There is a nonzero vector $\varphi \in L$ such that $|\varphi| \le 2^{(m-1)/2}(\det L)^{1/m}$. By Theorem 2.3, each qualifying value $r \in \mathbf{Q}$ is a root of φ. The degree of φ is at most $m - 1$ by construction of L, so there are at most $m - 1$ roots of φ. □

Theorem 2.6. *Let d, k, m be positive integers with $m \ge dk + 1$. Let $f \in \mathbf{Q}[x]$ be a polynomial of degree d with leading coefficient $f_d > 0$. Let $g \in \mathbf{Q}[x]$ be a polynomial of degree 1 with leading coefficient $g_1 > 0$. Assume that $g_1 < (g_1^d/f_d)^{2k/(m-1)-dk(k+1)/m(m-1)}/2m^{1/(m-1)}$. Then there are at most $m - 1$ values $r \in \mathbf{Q}$ such that $|r| \le 1$, $f(r) \in \mathbf{Z}$, and $g(r) \in \mathbf{Z}$.*

Take, for example, $k = 1$ and $m = 2d$: if $2g_1 < (g_1^d/2df_d)^{1/(2d-1)}$ then there are at most $2d - 1$ values $r \in \mathbf{Q}$ such that $|r| \le 1$, $f(r) \in \mathbf{Z}$, and $g(r) \in \mathbf{Z}$.

Proof. Apply lattice-basis reduction to Theorem 2.4. □

3. Parameter choice and other optimizations

This section discusses the choice of k amd m in Section 2, and other ways to speed up the computation of the desired numbers r.

The history of this subject—see Section 1—shows each application progressing from simple choices of k and m to near-optimal choices of k and m. It turns out to be possible to unify all of these application-specific optimizations into a few straightforward formulas: Theorem 3.2 states near-optimal choices of k and m for Theorem 2.3, and Theorem 3.4 states near-optimal choices of k and m for Theorem 2.4. Future applications should be able to reuse these unified theorems, rather than wasting time redoing the same optimizations from scratch.

Parameter choice for Theorem 2.3. Theorem 2.3 assumes that $\gcd\{1, f(r)\} > \gamma$, where $\gamma = m^{1/2k}(2g_1)^{(m-1)/2k}(g_1^d/f_d)^{d(k+1)/2m-1}$. How small can one make this lower bound γ by varying m and k?

Assume that g_1 and $1/f_d$ exceed 1. Theorem 3.1 then says that γ is smaller than $\beta = m^{1/2k}(2g_1)^{\alpha d(1+1/2k)}f_d/g_1^d$, where $\alpha = \sqrt{1 + (\lg(1/f_d))/\lg((2g_1)^d)}$, if m is chosen as $\lceil \alpha d(k+1) \rceil$. This choice of m approximately balances the factors $(2g_1)^{(m-1)/2k}$ and $(g_1^d/f_d)^{d(k+1)/2m}$ in Theorem 2.3. Note that $\alpha \ge 1$, so $m \ge dk+d$. Note also that m is not difficult to compute: comparing $\alpha d(k+1)$ to an integer boils down to comparing integer powers of f_d and $2g_1$.

As k increases (slowing down the computation of φ), β converges to $(2g_1)^{\alpha d} f_d/g_1^d$, which is very close to a lower bound on γ. The quantity $(2g_1)^{\alpha d}$ is the doubly-geometric average of $(2g_1)^d$ and $(2g_1)^d/f_d$. Theorem 3.2 considers the special case $k = \lceil \alpha d \lceil \lg 2g_1 \rceil /2 \rceil$, which balances the desire for a small β against the desire for small lattice ranks.

For comparison: If $k = 1$, the optimal choice of m is approximately $\sqrt{2}\alpha d$ for large αd, with $\gamma \approx (2g_1)^{\sqrt{2}\alpha d} f_d/g_1^d$. Allowing larger k thus changes the exponent of $2g_1$ by a factor of approximately $\sqrt{2}$.

Theorem 3.1. *Let $f \in \mathbf{Q}[x]$ be a polynomial of positive degree d with leading coefficient $f_d \in (0,1]$. Let $g \in \mathbf{Q}[x]$ be a polynomial of degree 1 with leading coefficient $g_1 \geq 1$. Let k be a positive integer. Define $\alpha = \sqrt{1 + (\lg(1/f_d))/\lg((2g_1)^d)}$, $m = \lceil \alpha d(k+1) \rceil$, $\beta = m^{1/2k}(2g_1)^{\alpha d(1+1/2k)} f_d/g_1^d$, and L as above. Let $\varphi \in L$ be a nonzero vector such that $|\varphi| \leq 2^{(m-1)/2}(\det L)^{1/m}$. If $r \in \mathbf{Q}$, $|r| \leq 1$, $\gcd\{1, f(r)\} \geq \beta$, and $g(r) \in \mathbf{Z}$, then $\varphi(r) = 0$.*

Proof. First $m - 1 \leq \alpha d(k+1)$ so $(2g_1)^{(m-1)/2k} \leq (2g_1)^{\alpha d(k+1)/2k}$. Second $1/m \leq 1/\alpha d(k+1)$ so $(g_1^d/f_d)^{d(k+1)/2m} \leq (g_1^d/f_d)^{1/2\alpha} < ((2g_1)^d/f_d)^{1/2\alpha} = (2g_1)^{\alpha d/2}$ by choice of α. Thus $m^{1/2k}(2g_1)^{(m-1)/2k}(g_1^d/f_d)^{d(k+1)/2m} f_d/g_1^d < \beta$. Apply Theorem 2.3. ☐

Theorem 3.2. *Let $f \in \mathbf{Q}[x]$ be a polynomial of positive degree d with leading coefficient $f_d \in (0,1]$. Let $g \in \mathbf{Q}[x]$ be a polynomial of degree 1 with leading coefficient $g_1 \geq 1$. Define $\alpha = \sqrt{1 + (\lg(1/f_d))/\lg((2g_1)^d)}$, $k = \lceil \alpha d \lceil \lg 2g_1 \rceil /2 \rceil$, $m = \lceil \alpha d(k+1) \rceil$, and L as above. Let $\varphi \in L$ be a nonzero vector such that $|\varphi| \leq 2^{(m-1)/2}(\det L)^{1/m}$. If $r \in \mathbf{Q}$, $|r| \leq 1$, $\gcd\{1, f(r)\} \geq 2m^{1/2k}(2g_1)^{\alpha d} f_d/g_1^d$, and $g(r) \in \mathbf{Z}$, then $\varphi(r) = 0$.*

Proof. By construction $k \geq \alpha d(\lg 2g_1)/2$, so $1 \geq (\lg 2g_1)\alpha d/2k$, so $2 \geq (2g_1)^{\alpha d/2k}$. Thus $\gcd\{1, f(r)\} \geq \beta$ where $\beta = m^{1/2k}(2g_1)^{\alpha d(1+1/2k)} f_d/g_1^d$. Apply Theorem 3.1. ☐

Parameter choice for Theorem 2.4. Theorem 2.4 assumes that g_1 is smaller than $(g_1^d/f_d)^{2k/(m-1)-dk(k+1)/m(m-1)}/2m^{1/(m-1)}$. How large can one make this exponent $2k/(m-1) - dk(k+1)/m(m-1)$ by varying m and k?

Theorem 3.3 chooses $m = dk + d$, achieving exponent $k/(dk + d - 1)$, which is reasonably close to optimal. As k increases (slowing down the computation of φ), the exponent converges to $1/d$. Theorem 3.4 considers the special case $k = \lceil \lceil \lg(g_1^d/2^d f_d) \rceil /d \rceil$, which balances the desire for a large exponent against the desire for small lattice ranks.

Theorem 3.3. *Let $f \in \mathbf{Q}[x]$ be a polynomial of positive degree d with leading coefficient $f_d > 0$. Let $g \in \mathbf{Q}[x]$ be a polynomial of degree 1 with leading coefficient $g_1 > 0$. Let k be a positive integer. Define $m = dk + d$ and $L = \mathbf{Z} + \mathbf{Z}g + \cdots + \mathbf{Z}g^{d-1} + \mathbf{Z}f + \mathbf{Z}gf + \cdots + \mathbf{Z}g^{d-1}f + \cdots + \mathbf{Z}f^k + \mathbf{Z}gf^k + \cdots + \mathbf{Z}g^{d-1}f^k$. Let $\varphi \in L$ be a nonzero vector such that $|\varphi| \leq 2^{(m-1)/2}(\det L)^{1/m}$. Assume that $g_1 < (g_1^d/f_d)^{k/(m-1)}/2m^{1/(m-1)}$. If $r \in \mathbf{Q}$, $|r| \leq 1$, $f(r) \in \mathbf{Z}$, and $g(r) \in \mathbf{Z}$, then $\varphi(r) = 0$.*

Proof. $d(k+1)/m = 1$ so $2k/(m-1) - dk(k+1)/m(m-1) = 2k/(m-1) - k/(m-1) = k/(m-1)$. Apply Theorem 2.4. ☐

Theorem 3.4. *Let $f \in \mathbf{Q}[x]$ be a polynomial of positive degree d with leading coefficient $f_d \in (0, 1/8^d)$. Let $g \in \mathbf{Q}[x]$ be a polynomial of degree 1 with leading coefficient $g_1 \ge 1/4$. Define $k = \lceil \lceil \lg(g_1^d/2^d f_d) \rceil / d \rceil$, $m = dk + d$, and $L = \mathbf{Z} + \mathbf{Z}g + \cdots + \mathbf{Z}g^{d-1} + \mathbf{Z}f + \mathbf{Z}gf + \cdots + \mathbf{Z}g^{d-1}f + \cdots + \mathbf{Z}f^k + \mathbf{Z}gf^k + \cdots + \mathbf{Z}g^{d-1}f^k$. Let $\varphi \in L$ be a nonzero vector such that $|\varphi| \le 2^{(m-1)/2}(\det L)^{1/m}$. If $r \in \mathbf{Q}$, $|r| \le 1$, $f(r) \in \mathbf{Z}$, and $g(r) \in \mathbf{Z}$, then $\varphi(r) = 0$.*

Proof. First $g_1^d/2^d f_d > (1/4)^d/(2/8)^d = 1$, so k is a positive integer. Next $m = d(k+1) \ge 2$, so $\lg m \le m - 1$, so $1 \le 2/m^{1/(m-1)}$. Next $m - 1 = dk + d - 1 \ge \lg(g_1^d/2^d f_d) + d - 1 \ge ((d-1)/d)\lg(g_1^d/2^d f_d) + d - 1 = ((m-1-dk)/d)\lg(g_1^d/f_d)$ so $d(m-1) \ge (m-1-dk)\lg(g_1^d/f_d)$ so $1 \ge (1/d - k/(m-1))\lg(g_1^d/f_d)$; i.e., $(g_1^d/f_d)^{1/d-k/(m-1)} \le 2$. Next $g_1 = (g_1^d/f_d)^{1/d-k/(m-1)}(g_1^d/f_d)^{k/(m-1)} f_d^{1/d}(1) < (2)(g_1^d/f_d)^{k/(m-1)}(1/8)(2/m^{1/(m-1)}) = (g_1^d/f_d)^{k/(m-1)}/2m^{1/(m-1)}$. Finally apply Theorem 3.3. □

Computation. Theorems 3.1, 3.2, 3.3, and 3.4, like Theorems 2.3 and 2.4, can easily be converted into algorithms to compute the set of r's. Theorems 3.5, 3.6, 3.7, and 3.8, like Theorems 2.5 and 2.6, measure the complexity of these algorithms by stating bounds on the output size.

Theorem 3.5. *Let $f \in \mathbf{Q}[x]$ be a polynomial of positive degree d with leading coefficient $f_d \in (0, 1]$. Let $g \in \mathbf{Q}[x]$ be a polynomial of degree 1 with leading coefficient $g_1 \ge 1$. Let k be a positive integer. Define $\alpha = \sqrt{1 + (\lg(1/f_d))/\lg((2g_1)^d)}$ and $m = \lceil \alpha d(k+1) \rceil$. Then there are at most $m - 1$ values $r \in \mathbf{Q}$ such that $|r| \le 1$, $\gcd\{1, f(r)\} \ge m^{1/2k}(2g_1)^{\alpha d(1+1/2k)} f_d/g_1^d$, and $g(r) \in \mathbf{Z}$.*

Proof. Apply lattice-basis reduction to Theorem 3.1. □

Theorem 3.6. *Let $f \in \mathbf{Q}[x]$ be a polynomial of positive degree d with leading coefficient $f_d \in (0, 1]$. Let $g \in \mathbf{Q}[x]$ be a polynomial of degree 1 with leading coefficient $g_1 \ge 1$. Define $\alpha = \sqrt{1 + (\lg(1/f_d))/\lg((2g_1)^d)}$, $k = \lceil \alpha d \lceil \lg 2g_1 \rceil / 2 \rceil$, and $m = \lceil \alpha d(k+1) \rceil$. Then there are at most $m-1$ values $r \in \mathbf{Q}$ such that $|r| \le 1$, $\gcd\{1, f(r)\} \ge 2m^{1/2k}(2g_1)^{\alpha d} f_d/g_1^d$, and $g(r) \in \mathbf{Z}$.*

The bound $m - 1$ is approximately $(\lg((2g_1)^d) + \lg(1/f_d))d/2$. The limit on $\gcd\{1, f(r)\}$ is approximately f_d/g_1^d times the doubly-geometric average of $(2g_1)^d$ and $(2g_1)^d/f_d$.

Proof. Apply lattice-basis reduction to Theorem 3.2. □

Theorem 3.7. *Let $f \in \mathbf{Q}[x]$ be a polynomial of positive degree d with leading coefficient $f_d > 0$. Let $g \in \mathbf{Q}[x]$ be a polynomial of degree 1 with leading coefficient $g_1 > 0$. Let k be a positive integer. Define $m = dk + d$. Assume that $g_1 < (g_1^d/f_d)^{k/(m-1)}/2m^{1/(m-1)}$. Then there are at most $m - 1$ values $r \in \mathbf{Q}$ such that $|r| \le 1$, $f(r) \in \mathbf{Z}$, and $g(r) \in \mathbf{Z}$.*

Proof. Apply lattice-basis reduction to Theorem 3.3. □

Theorem 3.8. *Let $f \in \mathbf{Q}[x]$ be a polynomial of positive degree d with leading coefficient $f_d \in (0, 1/8^d)$. Let $g \in \mathbf{Q}[x]$ be a polynomial of degree 1 with leading coefficient $g_1 \ge 1/4$. Then there are fewer than $\lg(g_1^d/f_d) + d - 1$ values $r \in \mathbf{Q}$ such that $|r| \le 1$, $f(r) \in \mathbf{Z}$, and $g(r) \in \mathbf{Z}$.*

Proof. Apply lattice-basis reduction to Theorem 3.4, using $m < \lg(g_1^d/f_d) + d$. □

Combining Theorem 3.3 with brute force. Theorem 3.3, applied to f and g, finds all rational numbers $r \in [-1,1]$ with $f(r), g(r) \in \mathbf{Z}$. The same theorem, applied to $f(x+2)$ and $g(x+2)$, finds all rational numbers $r \in [1,3]$ with $f(r), g(r) \in \mathbf{Z}$. With c such computations, involving c lattices of rank $m = dk + d$, one can cover an r interval of length $2c$.

One can view Theorem 3.3 as searching the rationals r with $g(r) \in \mathbf{Z}$, to see which ones also have $f(r) \in \mathbf{Z}$. In an interval of length $2c$, there are approximately $2cg_1 < c(g_1^d/f_d)^{k/(dk+d-1)}$ rationals r with $g(r) \in \mathbf{Z}$, so the number of r's searched per unit time is approximately $(g_1^d/f_d)^{k/(dk+d-1)}$ divided by the time to handle a lattice of rank $dk+d$. Given f and g, one can choose k to (approximately) maximize this ratio. This idea appears in [8].

Smaller improvements. Another way to expand the number of r's searched is to perform several rational-root calculations per lattice, searching for roots of shifts of φ. For example, the roots of $\varphi - 2, \varphi - 1, \varphi, \varphi + 1, \varphi + 2$ include all $r \in \mathbf{Q}$ such that $|r| \le 1$, $f(r) \in \mathbf{Z}$, and $g(r) \in \mathbf{Z}$, provided that $g_1 < 3(g_1^d/f_d)^{k/(m-1)}/2m^{1/(m-1)}$; note the 3 here. I learned this idea from Lenstra.

The choice of m in Theorem 3.1 is not exactly optimal. It is better to have the computer run through all pairs (k, m), in increasing order of the r computation time, until finding a pair (k, m) where the bound in Theorem 2.3 is satisfactory. Similar comments apply to Theorem 3.3.

I quoted lattice-basis reduction in Section 2 as producing nonzero vectors $\varphi \in L$ such that $|\varphi|$ is at most $2^{(m-1)/2}(\det L)^{1/m}$. Slower reduction algorithms can shrink the factor $2^{(m-1)/2}$; even without this extra work, lattice-basis reduction often produces a vector φ with $|\varphi| < (\det L)^{1/m}$. Bounds that depend on φ, as in Theorem 2.1, are slightly better than bounds that depend solely on $\det L$.

In Theorems 2.3, 2.4, 3.1, and 3.3, the lattice L can be replaced by a slightly smaller lattice, namely $\mathbf{Z} + \mathbf{Z}g + \mathbf{Z}g(g-1)/2 + \mathbf{Z}g(g-1)(g-2)/6 + \cdots$. The point is that $g(r)(g(r)-1)/2$ etc. are integers if $g(r)$ is an integer. This idea was published in [11], with credit to Howgrave-Graham and Lenstra independently.

A few years earlier, Howgrave-Graham in [20, Section 4.5.2] had made the similar observation that f could often be replaced by $f/d!$, after suitable tweaking of the coefficients of f.

Another slight improvement is to change the metric used to define the lattice, replacing $1, x, x^2, \ldots, x^{m-1}$ with Chebyshev polynomials. This idea was published by Coppersmith in [11, page 24], with partial credit (of unclear scope) to Boneh.

4. Example: roots mod n given their high bits

This section explains how to search an interval $[-H, H]$ for integer roots of an integer polynomial p modulo n, if H is not too large. For example, this section explains how to search the interval $[t - H, t + H]$ for cube roots of s modulo n, if H is not too large; here $p = (x + t)^3 - s$.

As in previous sections, the search method is parametrized by an exponent k. Theorem 4.2 uses a particular k that works well for most applications; Theorem 4.1 is more general and allows k to be tuned for the reader's application. The subsequent theorems in this section measure the cost of the resulting computations.

The choice of k in Theorem 4.2 allows H up to about $n^{1/d}$. For example, one can find cube roots of s modulo n in any interval of length about $n^{1/3}$. This generalizes the obvious fact that one can quickly compute r from $r^3 \bmod n$ if $0 \le r < n^{1/3}$.

For comparison, the simpler choice $k = 1$ allows H up to only about $n^{2/d(d+1)}$; for example, about $n^{1/6}$ for $d = 3$.

Theorem 4.1. *Let n be a positive integer. Let $p \in \mathbf{Z}[x]$ be a monic polynomial of positive degree d. Let k be a positive integer. Define $m = dk + d$. Let H be a positive integer smaller than $n^{k/(m-1)}/2m^{1/(m-1)}$. Define $f = p(Hx)/n \in \mathbf{Q}[x]$, $g = Hx \in \mathbf{Q}[x]$, and $L = \mathbf{Z} + \mathbf{Z}g + \cdots + \mathbf{Z}g^{d-1} + \mathbf{Z}f + \mathbf{Z}gf + \cdots + \mathbf{Z}g^{d-1}f + \cdots + \mathbf{Z}f^k + \mathbf{Z}gf^k + \cdots + \mathbf{Z}g^{d-1}f^k$. Let $\varphi \in L$ be a nonzero vector such that $|\varphi| \le 2^{(m-1)/2}(\det L)^{1/m}$. If $s \in \mathbf{Z}$, $p(s) \in n\mathbf{Z}$, and $|s| \le H$, then $\varphi(s/H) = 0$.*

Proof. Define $r = s/H$. By hypothesis $r \in \mathbf{Q}$, $|r| \le 1$, $f(r) = p(s)/n \in \mathbf{Z}$, $g(r) = s \in \mathbf{Z}$, and $g_1 = H < n^{k/(m-1)}/2m^{1/(m-1)} = (g_1^d/f_d)^{k/(m-1)}/2m^{1/(m-1)}$. Apply Theorem 3.3. □

Theorem 4.2. *Let n be a positive integer. Let $p \in \mathbf{Z}[x]$ be a monic polynomial of positive degree d. Let H be a positive integer smaller than $n^{1/d}/8$. Define $k = \lceil (\lg n)/d \rceil - 1$ and $m = dk + d$. Define $f = p(Hx)/n \in \mathbf{Q}[x]$, $g = Hx \in \mathbf{Q}[x]$, and $L = \mathbf{Z}+\mathbf{Z}g+\cdots+\mathbf{Z}g^{d-1}+\mathbf{Z}f+\mathbf{Z}gf+\cdots+\mathbf{Z}g^{d-1}f+\cdots+\mathbf{Z}f^k+\mathbf{Z}gf^k+\cdots+\mathbf{Z}g^{d-1}f^k$. Let $\varphi \in L$ be a nonzero vector such that $|\varphi| \le 2^{(m-1)/2}(\det L)^{1/m}$. If $s \in \mathbf{Z}$, $p(s) \in n\mathbf{Z}$, and $|s| \le H$, then $\varphi(s/H) = 0$.*

Proof. The leading coefficient f_d of f is $H^d/n \in (0, 1/8^d)$. The leading coefficient g_1 of g is $H > 1/4$. The quotient $g_1^d/2^d f_d$ is $H^d/2^d(H^d/n) = n/2^d$. Consequently $k = \lceil (\lg n - d)/d \rceil = \lceil \lceil \lg n - d \rceil /d \rceil = \lceil \lceil \lg(g_1^d/2^d f_d) \rceil /d \rceil$.

Define $r = s/H$. By hypothesis $r \in \mathbf{Q}$, $|r| \le 1$, $f(r) = p(s)/n \in \mathbf{Z}$, and $g(r) = s \in \mathbf{Z}$. Apply Theorem 3.4. □

Theorem 4.3. *Let n be a positive integer. Let $p \in \mathbf{Z}[x]$ be a monic polynomial of positive degree d. Let k be a positive integer. Define $m = dk + d$. Let H be a positive integer smaller than $n^{k/(m-1)}/2m^{1/(m-1)}$. Then there are at most $m - 1$ integers $s \in \{-H, -H+1, \dots, -1, 0, 1, \dots, H-1, H\}$ such that $p(s) \in n\mathbf{Z}$.*

Proof. Apply lattice-basis reduction to Theorem 4.1. □

Theorem 4.4. *Let n be a positive integer. Let $p \in \mathbf{Z}[x]$ be a monic polynomial of positive degree d. Let H be a positive integer smaller than $n^{1/d}/8$. Then there are fewer than $\lg n + d - 1$ integers $s \in \{-H, -H+1, \dots, -1, 0, 1, \dots, H-1, H\}$ such that $p(s) \in n\mathbf{Z}$.*

Proof. Apply lattice-basis reduction to Theorem 4.2, using $m < \lg n + d$. □

History. As indicated in Section 1, the $n^{2/d(d+1)}$ result was first published by Håstad, and the $n^{1/d}$ result was first published by Coppersmith. Both authors used their results to break various naive forms of the RSA cryptosystem.

The results also have a positive application to cryptography: viz., compressing RSA (or Rabin) signatures. Instead of transmitting a cube root (or square root) of s modulo n, one can transmit the top 2/3 (or 1/2) of the bits of the root. However, this application is now obsolete, because Bleichenbacher in [4] proposed a different compression mechanism allowing substantially faster decompression and verification: compress the cube root to an integer v such that the remainder $v^3 s$ mod n is a cube in $\mathbf{Z}$.

A numerical example. Define $n = 2844847044114666594769924451263$. How do we find, near the integer $1249180057712313741000000000000$, a square root of $1982518464324230691670577165029$ modulo n? In other words: Define $p = (x + 1249180057712313741000000000000)^2 - 1982518464324230691670577165029$. How do we find a small root of p modulo n?

Choose $k = 2$ and $H = 10^{12}/2$. Define $d = \deg p = 2$ and $m = dk + d = 6$. Then $m(2H)^{m-1} = 6 \cdot 10^{60} < n^2$ so $H < n^{k/(m-1)}/2m^{1/(m-1)}$. Define $f = p(Hx)/n$, $g = Hx$, and $L = \mathbf{Z} + \mathbf{Z}g + \mathbf{Z}f + \mathbf{Z}gf + \mathbf{Z}f^2 + \mathbf{Z}gf^2$.

Reduce the basis $1, g, f, gf, f^2, gf^2$ to find a nonzero vector in L of length at most $2^{(m-1)/2}(\det L)^{1/m} = 2^{5/2}H^{5/2}/n \approx 0.352$: for example, the vector $\varphi = 3gf^2 - 1499016069254776489264474669541 4f^2 + 1645555060488421911465440990695 3gf - 7073107916020226404213966825942 25363949260f + (\cdots)g + (\cdots)1 =$

$$\begin{aligned}
&(93750000000000000000000000000000000000000000000000000000000/n^2)x^5\\
&- (40296668463375000000000000000000000000000000000000000000000/n^2)x^4\\
&- (8521407708770506207316875000000000000000000000000000000000/n^2)x^3\\
&+ (7549559148957274134432151119009658000000000000000000000000/n^2)x^2\\
&+ (85256085569824577108175046901012420957509825119500000000000/n^2)x\\
&- (73391645786690147620682490399407175727933183364776412308271/n^2)1,
\end{aligned}$$

of length approximately 0.019.

The only rational root of φ is $372834385559/H$. Check that $p(372834385559)$ is a multiple of n, i.e., that $1249180057712313741372834385559$ is a square root of $1982518464324230691670577165029$ modulo n.

Theorem 4.1 guaranteed that this procedure would find every integer root of p modulo n in the interval $[-H, H]$. (Theorem 2.1 guaranteed an even wider interval after $|\varphi|$ turned out to be noticeably smaller than $2^{(m-1)/2}(\det L)^{1/m}$.) This is much faster than separately checking each of the $10^{12} + 1$ integers in this interval.

5. Example: constrained divisors of n

This section explains how to search for small integers s such that

- $u + s$ divides n; or, more generally,
- $u + vs$ divides n, where v is coprime to n; or, more generally,
- $(u + vs)^d$ divides n, where v is coprime to n.

For example, by choosing $d = 1$ and choosing v as a large power of 2, one can search for divisors of n having specified low bits.

As in previous sections, the search method is parametrized by an exponent k. Theorem 5.2 uses a particular k that works well for most applications; Theorem 5.1 is more general and allows k to be tuned for the reader's application. The remaining theorems in this section measure the cost of the resulting computations.

Section 6 combines this search method with brute force to search a somewhat wider range of s. Conclusion in a nutshell: if $v \geq n^{1/4}$, and v is coprime to n, then one can quickly find all divisors of n in $(u + v\mathbf{Z}) \cap [1, n^{1/2}]$.

Theorem 5.1. *Let d, n, u, v, w, H be positive integers such that $vw - 1 \in n\mathbf{Z}$ and $n \geq H^d$. Let k be a positive integer. Define $\alpha = \sqrt{(\lg 2^d n)/\lg 2^d H^d}$, $m = \lceil \alpha d(k+1) \rceil$, $f = (uw + Hx)^d/n \in \mathbf{Q}[x]$, $g = Hx \in \mathbf{Q}[x]$, $\lambda = m^{1/2kd}(2H)^{\alpha(1+1/2k)}$, and L as above. Let $\varphi \in L$ be a nonzero vector such that $|\varphi| \leq 2^{(m-1)/2}(\det L)^{1/m}$. If $s \in \mathbf{Z}$, $|s| \leq H$, $u + vs \geq \lambda$, and $n \in (u + vs)^d\mathbf{Z}$, then $\varphi(s/H) = 0$.*

The polynomial $(uw + Hx)^d/n$ used here is better than $(u + vHx)^d/n$ when $v > 1$: it has a smaller leading coefficient, so it produces a smaller lattice L.

Proof. By hypothesis $u + vs \geq \lambda > 0$. Note that $u + vs$ divides $uw + s$. Indeed, $u + vs$ divides $(u + vs)w = uw + s + (vw - 1)s$; but $u + vs$ also divides $(u + vs)^d$, hence n, hence $vw - 1$.

Define $r = s/H$. Then $f(r) = (uw + s)^d/n$. The numerator $(uw + s)^d$ and the denominator n are both divisible by $(u + vs)^d$, so $\gcd\{1, f(r)\} \geq (u + vs)^d/n \geq \lambda^d/n = m^{1/2k}(2H)^{\alpha d(1+1/2k)}/n$.

By hypothesis $g_1 = H \geq 1$; $1/f_d = n/H^d \geq 1$; $\alpha = \sqrt{1 + \lg(1/f_d)/\lg((2g_1)^d)}$; $r \in \mathbf{Q}$; $|r| = |s|/H \leq 1$; $\gcd\{1, f(r)\} \geq m^{1/2k}(2g_1)^{\alpha d(1+1/2k)} f_d/g_1^d$; and $g(r) = s \in \mathbf{Z}$. Apply Theorem 3.1. □

Theorem 5.2. *Let d, n, u, v, w, H be positive integers such that $vw - 1 \in n\mathbf{Z}$ and $n \geq H^d$. Define $\alpha = \sqrt{(\lg 2^d n)/\lg 2^d H^d}$, $k = \lceil \alpha d \lceil \lg 2H \rceil /2 \rceil$, $m = \lceil \alpha d(k + 1) \rceil$, $f = (uw + Hx)^d/n \in \mathbf{Q}[x]$, $g = Hx \in \mathbf{Q}[x]$, and L as above. Let $\varphi \in L$ be a nonzero vector such that $|\varphi| \leq 2^{(m-1)/2}(\det L)^{1/m}$. If $s \in \mathbf{Z}$, $|s| \leq H$, $u + vs \geq 2^{1/d} m^{1/2kd}(2H)^{\alpha}$, and $n \in (u + vs)^d\mathbf{Z}$, then $\varphi(s/H) = 0$.*

The lattice rank m here is larger than $(d/2)\lg 2^d n$. It is only slightly larger for typical values of d, n, H.

Proof. By hypothesis $2H \geq 2$ so $\lg 2H \geq 1$; hence k is a positive integer. Also $2k \geq \alpha d \lg 2H$ so $2^{1/d} \geq (2H)^{\alpha/2k}$ so $u + vs \geq \lambda$ where $\lambda = m^{1/2kd}(2H)^{\alpha(1+1/2k)}$. Apply Theorem 5.1. □

Theorem 5.3. *Let d, n, u, v, H be positive integers such that $\gcd\{v, n\} = 1$ and $n \geq H^d$. Let k be a positive integer. Define $\alpha = \sqrt{(\lg 2^d n)/\lg 2^d H^d}$ and $m = \lceil \alpha d(k + 1) \rceil$. Then there are at most $m - 1$ integers $s \in \{-H, \ldots, -1, 0, 1, \ldots, H\}$ such that $u + vs \geq m^{1/2kd}(2H)^{\alpha(1+1/2k)}$ and $n \in (u + vs)^d\mathbf{Z}$.*

Proof. Find a positive integer w with $vw - 1 \in n\mathbf{Z}$. Apply lattice-basis reduction to Theorem 5.1. □

Theorem 5.4. *Let d, n, u, v, H be positive integers such that $\gcd\{v, n\} = 1$ and $n \geq H^d$. Define $\alpha = \sqrt{(\lg 2^d n)/\lg 2^d H^d}$, $k = \lceil \alpha d \lceil \lg 2H \rceil /2 \rceil$, and $m = \lceil \alpha d(k + 1) \rceil$. Then there are at most $m - 1$ integers $s \in \{-H, \ldots, -1, 0, 1, \ldots, H\}$ such that $u + vs \geq 2^{1/d} m^{1/2kd}(2H)^{\alpha}$ and $n \in (u + vs)^d\mathbf{Z}$.*

Proof. Find a positive integer w with $vw - 1 \in n\mathbf{Z}$. Apply lattice-basis reduction to Theorem 5.2. □

History. As indicated in Section 1, results of this type were developed in two contexts independently. The first context is proving primality of n: the Adleman-Pomerance-Rumely method in [3] exhibits some arithmetic progressions and proves, using factors of unit groups of extensions of $\mathbf{Z}/n$, that every divisor of n is in one of those progressions. The second context is factoring an RSA public key n given part of the secret key: for example, finding a divisor of n given the low bits of the divisor.

In the first context, Lenstra in [24] showed how to find all divisors of n in an arithmetic progression $u + v\mathbf{Z}$ with $\lg v > (1/3)\lg n$. Konyagin and Pomerance in [22, Algorithm 3.2] improved $(1/3)\lg n$ to $0.3\lg n$, in the special case $u = 1$. This

$0.3 \lg n$ result, for any u, follows from Theorem 2.3 with $m = 5$ and $k = 2$; I have not checked whether the resulting algorithm is equivalent to the Konyagin-Pomerance algorithm.

In the second context, Rivest and Shamir in [31] gave a heuristic outline of a method to find a divisor of n given about $(1/3) \lg n$ high bits of the divisor. Coppersmith in [9] proved that a much more complicated bivariate algorithm would find a divisor of n given $(0.25 + \epsilon) \lg n$ high bits of the divisor. Howgrave-Graham in [19] achieved $(0.25+\epsilon) \lg n$ with the simpler algorithm shown here. Each of these authors commented that the method also applied to low bits, but they did not generalize to other arithmetic progressions.

These two threads in the literature were finally combined in [20, Section 5.5] and [12]: Coppersmith, Howgrave-Graham, and Nagaraj improved the Konyagin-Pomerance $0.3 \lg n$ to $(0.25 + \epsilon) \lg n$. Lenstra subsequently pointed out that the ϵ could be eliminated; see Section 6 for further discussion.

Boneh, Durfee, and Howgrave-Graham in [7] pointed out, at least for $v = 1$, the further generalization from divisors $u + vs$ to divisors $(u + vs)^d$. As d increases, the allowable range of H shrinks, but the range of interesting divisors shrinks more quickly. At an extreme, for d larger than about $\sqrt{\lg n}$, this method finds d-power divisors of n more quickly than the elliptic-curve method.

A numerical example. Consider the problem of finding $p \approx 1814430925000000$ such that p^2 divides $n = 3767375198243112483228974667456105955144630367$.

Define $d = 2$, $u = 1814430925000000$, $v = 1$, $w = 1$, $k = 2$, and $H = 10^6$. Define $\alpha = \sqrt{(\lg 4n)/\lg 4H^2} \approx 1.91424$ and $m = \lceil \alpha d(k + 1)\rceil = 12$. Then $u-H \geq \lambda$ where $\lambda = m^{1/2kd}(2H)^{\alpha(1+1/2k)}$. Define $f = (uw + Hx)^d/n = (u + Hx)^2/n$, $g = Hx$, and $L = \mathbf{Z}+\mathbf{Z}g+\mathbf{Z}f+\mathbf{Z}gf+\mathbf{Z}f^2+\mathbf{Z}gf^2+\mathbf{Z}g^2f^2+\mathbf{Z}g^3f^2+\mathbf{Z}g^4f^2+\mathbf{Z}g^5f^2+\mathbf{Z}g^6f^2+\mathbf{Z}g^7f^2$.

Reduce the basis $1, g, f, gf, f^2, gf^2, g^2f^2, g^3f^2, g^4f^2, g^5f^2, g^6f^2, g^7f^2$ to find a nonzero vector in L of length at most $2^{(m-1)/2}(\det L)^{1/m}$: for example, the vector

$$\begin{aligned}
&865428592905169853673115657973973290925440337012446696387011830651 6f^2\\
&\quad - 605010944490473289396767060950297824232645734932035 4f\\
&\quad - 272554120187872958477221635550721744176289110113680 5gf^2\\
&\quad - 13217375993392331719811049580402472 84\\
&\quad - 66688782294722083128266006947724553 32gf\\
&\quad + 75107328789962927234041809267291654 6g^2f^2\\
&\quad - 8325239807480528922 74g\\
&\quad - 1655777086232787858 39g^3f^2\\
&\quad + 22814g^4f^2,
\end{aligned}$$

of length approximately $2.3 \cdot 10^{-38}$. The only rational root of this polynomial is $339897/H$. Check that 1814430925339897^2 is a divisor of n.

Theorem 5.1 guaranteed that this procedure would find all divisors $(u+s)^2$ of n with $-H \leq s \leq H$. In fact, Theorem 2.3 guaranteed that $k = 2$ and $m = 7$ would have done the same job, and that $k = 1$ and $m = 5$ would have worked for the smaller interval $-450000 \leq s \leq 450000$.

6. Partitioning an arithmetic progression

Consider the problem of finding all divisors of n in $(u+v\mathbf{Z}) \cap [1, n^{1/2}]$. Here u, v, n are positive integers with $v \ge n^{1/4}$ and $\gcd\{v, n\} = 1$.

One can use Theorem 5.2 to find all divisors of n in the arithmetic progression $u - vH, u - v(H-1), \ldots, u + v(H-1), u + vH$. But there is a limitation here: the smallest entry $u - vH$ must exceed $2m^{1/2k}(2H)^\alpha$, approximately the doubly-geometric average of n and H^d. Another way to view the lower bound on $u - vH$ is as follows: if the smallest entry $u - vH$ is approximately $n^{1/\alpha}$ then the number of entries is limited to approximately n^{1/α^2}. In particular, if this method is searching for divisors around $n^{1/2}$, then it will search at most about $n^{1/4}$ entries in a specified arithmetic progression.

This might not sound like a serious limitation: by hypothesis $v \ge n^{1/4}$, so there are at most $n^{1/4}$ elements of $(u+v\mathbf{Z}) \cap [1, n^{1/2}]$. But one cannot search $n^{1/4}$ elements unless the *smallest* element searched is close to $n^{1/2}$.

The point of this section is that one can cover $(u+v\mathbf{Z}) \cap [1, n^{1/2}]$ with $O((\lg n)^{1/2})$ arithmetic progressions and $O((\lg n)^{1/2})$ extra integers, where each progression meets the conditions of Theorem 5.2. Consequently, one can quickly find all the divisors of n in $(u+v\mathbf{Z}) \cap [1, n^{1/2}]$. See Theorem 6.4 for a bound on the cost of this computation.

My bounds here are completely explicit. Various constants can be improved; my goal in selecting constants was not to obtain optimal cost bounds, but to simplify the statements and the proofs as far as possible while still achieving $O((\lg n)^{1/2})$.

Theorem 6.1. *Let n be an integer with $n \ge 2^{24}$. Let v be a positive integer with $\gcd\{v, n\} = 1$. Let H be an integer with $2 \le H \le n$. Define $\alpha = \sqrt{(\lg 2n)/\lg 2H}$. Let z be an integer with $z \ge 4(2H)^\alpha$. Then there are at most $2 \lg 2n + \sqrt{\lg 2n}$ divisors of n in $\{z, z+v, z+2v, \ldots, z+2vH\}$.*

Proof. The difference $2^{r\sqrt{2}} - 4r^2 - 2r$ is positive for all real numbers $r \ge 5$: its value at $r = 5$ is $2^{5\sqrt{2}} - 100 - 10 > 2^7 - 110 > 0$; its derivative at $r = 5$ is $2^{5\sqrt{2}}\sqrt{2}\log 2 - 40 - 2 > 0$; and its second derivative is $2^{r\sqrt{2}}(\sqrt{2}\log 2)^2 - 8 > 0$ for $r \ge 5$. In particular, $\sqrt{\lg 2n} \ge \sqrt{25} = 5$, so $2^{\sqrt{2\lg 2n}} \ge 4\lg 2n + 2\sqrt{\lg 2n}$.

Define $k = \lceil \alpha \lceil \lg 2H \rceil / 2 \rceil$. By hypothesis $H \ge 2$ so $\lg 2H \ge \lg 4 = 2$ so $2k \ge \alpha \lg 2H = \sqrt{(\lg 2n)\lg 2H} \ge \sqrt{2\lg 2n}$. Furthermore $H \le n$ so $\alpha \ge 1$ so $\alpha \lceil \lg 2H \rceil / 2 \ge 1$ so $k \le 2\alpha \lceil \lg 2H \rceil / 2 = \alpha \lceil \lg 2H \rceil \le 2\alpha \lg 2H$ so $\alpha(k+1) \le 2\alpha^2 \lg 2H + \alpha = 2\lg 2n + \alpha \le 2\lg 2n + \sqrt{\lg 2n}$.

Define $m = \lceil \alpha(k+1) \rceil$. Again $\alpha \ge 1$ so $\alpha(k+1) \ge 1$; thus $m \le 2\alpha(k+1) \le 4\lg 2n + 2\sqrt{\lg 2n} \le 2^{\sqrt{2\lg 2n}} \le 2^{2k}$. Consequently $z \ge 2m^{1/2k}(2H)^\alpha$.

Define $d = 1$ and $u = z + vH$. By Theorem 5.4, there are at most $m - 1$ divisors of n in $\{u - vH, \ldots, u - v, u, u + v, \ldots, u + vH\} \cap [2m^{1/2k}(2H)^\alpha, \infty) = \{z, z+v, z+2v, \ldots, z+2vH\}$. Finally $m - 1 \le \alpha(k+1) \le 2\lg 2n + \sqrt{\lg 2n}$. ☐

Theorem 6.2. *Let n, u, v be integers with $v \ge n^{1/4} \ge 2^{64}$ and $\gcd\{v, n\} = 1$. Let i be an integer with $8 \le i \le \sqrt{\lg n}/2$. Then there are at most $2\lg 2n + \sqrt{\lg 2n}$ divisors of n in $(u + v\mathbf{Z}) \cap [n^{1/2-2/i}, n^{1/2-2/(i+1)}]$.*

Proof. Define $H = \lfloor n^{1/4-2/(i+1)}/2 \rfloor$. Note that $n^{1/4-2/(i+1)} \ge n^{1/4-2/9} = n^{1/36} \ge 4$, so $H \ge 2$; and $H \le n^{1/4-2/(i+1)} \le n$. Define $\alpha = \sqrt{(\lg 2n)/\lg 2H}$. Define

z as the smallest element of $(u+v\mathbf{Z})\cap[n^{1/2-2/i},\infty)$. Note that $z \ge n^{1/2-2/i} \ge n^{1/2-2/8} = n^{1/4} \ge 2^{64}$.

I claim that $z+2Hv+v > n^{1/2-2/(i+1)}$. Proof: $H+1 > n^{1/4-2/(i+1)}/2$, so $z+2Hv+v \ge (1+2H+1)n^{1/4} > n^{1/4-2/(i+1)}n^{1/4} = n^{1/2-2/(i+1)}$.

I also claim that $z \ge 4(2H)^\alpha$. Proof: $i^2 \ge (i+1)(i-1)$; so $2/(i+1) \ge 2(i-1)/i^2$; so $2/(i+1)-2(i-2)/i^2 \ge 2/i^2 \ge 2/(\sqrt{\lg n}/2)^2 = 8/\lg n$; so

$$\begin{aligned}&\left(\left(\frac{1}{2}-\frac{2}{i}\right)\lg n-2\right)^2-\left(\frac{1}{4}-\frac{2}{i+1}\right)(\lg 2n)\lg n\\ &\quad=\left(\frac{2}{i+1}-\frac{2(i-2)}{i^2}\right)(\lg n)^2-\left(4\left(\frac{1}{2}-\frac{2}{i}\right)+\left(\frac{1}{4}-\frac{2}{i+1}\right)\right)\lg n+4\\ &\quad\ge\frac{8}{\lg n}(\lg n)^2-\left(4\left(\frac{1}{2}\right)+\left(\frac{1}{4}\right)\right)\lg n+4=\frac{23}{4}\lg n+4\ge 0;\end{aligned}$$

so $\alpha^2(\lg 2H)^2 = \lg 2n \lg 2H \le (\lg 2n)(1/4-2/(i+1))\lg n \le ((1/2-2/i)\lg n-2)^2$; so $\alpha\lg 2H \le |(1/2-2/i)\lg n-2| = (1/2-2/i)\lg n-2 \le \lg z-2$.

By Theorem 6.1, $\{z, z+v, \ldots, z+2vH\}$ has at most $2\lg 2n+\sqrt{\lg 2n}$ divisors of n. Finally $(u+v\mathbf{Z})\cap[n^{1/2-2/i}, n^{1/2-2/(i+1)}] \subseteq \{z, z+v, \ldots, z+2vH\}$. □

Theorem 6.3. *Let n, u, v be integers with $v \ge n^{1/4} \ge 2^{75}$ and $\gcd\{v,n\}=1$. Let i be an integer with $1 \le i \le \lceil 16\sqrt{\lg n}\rceil$. Then there are at most $2\lg 2n+\sqrt{\lg 2n}+1$ divisors of n in $(u+v\mathbf{Z})\cap[n^{1/2}/2^{i/4}, n^{1/2}/2^{(i-1)/4}]$.*

Proof. Define $H = \lfloor n^{1/4}/2^{(i+13)/4}\rfloor$. Note that $(\sqrt{\lg n}-8)^2 \ge (\sqrt{4\cdot 75}-8)^2 \ge 82$, so $\lg n-16\sqrt{\lg n} \ge 82-8^2 = 18$, so $\lg n - i \ge 17$, so $n^{1/4}/2^{(i+13)/4} = 2^{(\lg n-i-13)/4} \ge 2^{4/4} = 2$, so $H \ge 2$; and $H \le n^{1/4} \le n$. Define $\alpha = \sqrt{(\lg 2n)/\lg 2H}$. Define z as the smallest element of $(u+v\mathbf{Z})\cap[n^{1/2}/2^{i/4},\infty)$.

I claim that $z+2Hv+2v > n^{1/2}/2^{(i-1)/4}$. Proof: $H+1 > n^{1/4}/2^{(i+13)/4}$, and $1+2^{-9/4} \ge 2^{1/4}$, so $z+2(H+1)v > n^{1/2}/2^{i/4}+2^{-9/4}n^{1/2}/2^{i/4} \ge n^{1/2}/2^{(i-1)/4}$.

I claim that $z \ge 4(2H)^\alpha$. Proof: $((1/2)\lg n-i/4-2)^2-(\lg 2n)(\lg n-i-9)/4 = (i/4+2)^2+(i+9)/4 \ge 0$; so $\alpha^2(\lg 2H)^2 = (\lg 2n)\lg 2H \le (\lg 2n)(\lg n-i-9)/4 \le ((1/2)\lg n-i/4-2)^2$; and $(1/2)\lg n-i/4-2 \ge (\lg n-i)/4-2 \ge 17/4-2 \ge 0$, so $\alpha\lg 2H \le |(1/2)\lg n-i/4-2| = (1/2)\lg n-i/4-2 \le \lg z-2$.

By Theorem 6.1, $\{z, z+v, \ldots, z+2vH\}$ has at most $2\lg 2n+\sqrt{\lg 2n}$ divisors of n, so $\{z, z+v, \ldots, z+2vH+v\}$ has at most $2\lg 2n+\sqrt{\lg 2n}+1$ divisors of n. Finally $(u+v\mathbf{Z})\cap[n^{1/2}/2^{i/4}, n^{1/2}/2^{(i-1)/4}] \subseteq \{z, z+v, \ldots, z+2Hv+v\}$. □

Theorem 6.4. *Let n, u, v be integers with $v \ge n^{1/4} \ge 2^{75}$ and $\gcd\{v,n\}=1$. Define $\ell = \lg 2n$. Then there are at most $33\ell^{1.5}+4.5\ell+10\ell^{0.5}+2$ divisors of n in $(u+v\mathbf{Z})\cap[1,n^{1/2}]$.*

Proof. There is at most one divisor of n in $(u+v\mathbf{Z})\cap[1,n^{1/4}]$, since $v \ge n^{1/4}$.

Write $s = \lfloor\sqrt{\lg n}/2\rfloor$. Then $s \ge 8$. Also $s+1 > \sqrt{\lg n}/2$, so $n^{1/2-2/(s+1)} > n^{1/2-4/\sqrt{\lg n}}$. Apply Theorem 6.2 for each $i \in \{8, 9, \ldots, s\}$ to cover the intervals $[n^{1/2-2/8}, n^{1/2-2/9}]$, $[n^{1/2-2/9}, n^{1/2-2/10}]$, …, $[n^{1/2-2/s}, n^{1/2-2/(s+1)}]$: there are at most $(s-7)(2\ell+\ell^{0.5})$ divisors of n in $(u+v\mathbf{Z})\cap[n^{1/2-2/8}, n^{1/2-2/(s+1)}] \supseteq (u+v\mathbf{Z})\cap[n^{1/4}, n^{1/2-4/\sqrt{\lg n}}]$.

Write $t = \lceil 16\sqrt{\lg n}\rceil$. Then $t/4 \ge 4\sqrt{\lg n} = (4/\sqrt{\lg n})\lg n$, so $n^{1/2}/2^{t/4} \le n^{1/2-4/\sqrt{\lg n}}$. Apply Theorem 6.3 for each $i \in \{1, 2, \ldots, t\}$ to cover the intervals $[n^{1/2}/2^{1/4}, n^{1/2}/2^{0/4}]$, $[n^{1/2}/2^{2/4}, n^{1/2}/2^{1/4}]$, …, $[n^{1/2}/2^{t/4}, n^{1/2}/2^{(t-1)/4}]$: there are at most $t(2\ell + \ell^{0.5} + 1)$ divisors of n in $(u + v\mathbf{Z}) \cap [n^{1/2}/2^{t/4}, n^{1/2}/2^{0/4}] \supseteq (u + v\mathbf{Z}) \cap [n^{1/2-4/\sqrt{\lg n}}, n^{1/2}]$.

Add all of these bounds: there are at most $1+(s-7)(2\ell+\ell^{0.5})+t(2\ell+\ell^{0.5}+1) \le 1 + (\ell^{0.5}/2 - 7)(2\ell + \ell^{0.5}) + (1 + 16\ell^{0.5})(2\ell + \ell^{0.5} + 1) = 33\ell^{1.5} + 4.5\ell + 10\ell^{0.5} + 2$ divisors of n in $(u + v\mathbf{Z}) \cap [1, n^{1/2}]$. □

History. Coppersmith, Howgrave-Graham, and Nagaraj in [20, Section 5.5] and [12] explained how to construct lattices of total rank $O(\epsilon^{-3/2})$ that would handle all $v \ge n^{1/4+\epsilon}$ for all sufficiently large n. It is not clear whether one can take $\epsilon \approx 1/\lg n$ here: Coppersmith, Howgrave-Graham, and Nagaraj did not give simple formulas for their partition of $[1/4, 1/2]$ as a function of ϵ, and did not quantify "sufficiently large" as a function of ϵ.

Lenstra constructed lattices of total rank $O((\lg n)^2)$ handling all $v \ge n^{1/4}$, and asked whether one could achieve $O((\lg n)^{3/2})$. I constructed $O((\lg n)^{1/2})$ lattices of total rank $O((\lg n)^{3/2})$ handling all $v \ge n^{1/4}$; see Theorem 6.4.

The essential difference between the Coppersmith-Howgrave-Graham-Nagaraj proof and Lenstra's proof is in the analysis of how much progress is made by a $(2H + 1)$-entry arithmetic progression starting at z. The Coppersmith-Howgrave-Graham-Nagaraj proof has an advantage in handling small divisors: it chooses H much larger than z/v, producing a large lower bound on $\lg 2Hv$ and thus on $\lg(z + 2Hv)$, as in Theorem 6.2 here. Lenstra's proof has an advantage in handling large divisors: it allows H to be as small as, e.g., $0.1z/v$, and then observes that $\lg(z + 2Hv) \ge \lg 1.2z > \lg z + 0.25$, as in Theorem 6.3 here. My proof combines these advantages, and does some extra work to make all the bounds explicit.

Coppersmith, Howgrave-Graham, and Nagaraj tuned their choices of (k, m) more tightly than I have done, and they computed particularly good partitions (at least for the number-of-outputs cost measure) for several specific values of ϵ. As usual, I am leaving this level of optimization to the reader.

7. Example: codeword errors past half the minimum distance

Fix a positive integer H. Fix finitely many distinct primes $p_1, p_2, \ldots$. Assume that the product $n = p_1 p_2 \cdots$ is much larger than H. The **residue representation** of an integer $s \in [-H, H]$ is, by definition, the vector $(s \bmod p_1, s \bmod p_2, \ldots)$.

If $s' \ne s$ then there must be many differences between the residue representations of s and s'. Define the **distance** between s and s' as the sum of $\lg p_i$ for all i such that $s \bmod p_i \ne s' \bmod p_i$. Then the distance between s and s' is exactly $\lg n - \lg \gcd\{s' - s, n\}$, which is at least $\lg n - \lg 2H$ since $\gcd\{s' - s, n\} \le 2H$.

Thus the residue representation can tolerate some errors. For any vector v, there is at most one s whose representation has distance $< (\lg n - \lg 2H)/2$ from v.

This section explains how to efficiently recover s from a vector at any distance up to about $\lg n - \sqrt{(\lg 2n) \lg 2H}$. One first interpolates the vector into an integer $u \in \{0, 1, \ldots, n - 1\}$, and then finds s such that $\gcd\{u - s, n\}$ is large. For distances above $(\lg n - \lg 2H)/2$, there might be several possibilities for s; this section explains how to find them all.

Theorem 7.2 uses a particular parameter k that works well for most applications; Theorem 7.4 measures the cost of the resulting computation. Theorem 7.1 is more general and allows k to be tuned for the reader's application; Theorem 7.3 measures the cost of the resulting computation. The simplest case $k = 1$, $m = 2$ of Theorem

7.1 finds all s with $\gcd\{u-s,n\} > (4Hn)^{1/2}$, i.e., with distance smaller than $(\lg n - \lg 4H)/2$; there is at most one such s.

Theorem 7.1. *Let n, u, H be positive integers such that $n \ge H$. Let k be a positive integer. Define $\alpha = \sqrt{(\lg 2n)/\lg 2H}$, $m = \lceil \alpha(k+1) \rceil$, $\lambda = m^{1/2k}(2H)^{\alpha(1+1/2k)}$, $f = (Hx-u)/n \in \mathbf{Q}[x]$, $g = Hx \in \mathbf{Q}[x]$, $d = 1$, and L as above. Let $\varphi \in L$ be a nonzero vector such that $|\varphi| \le 2^{(m-1)/2}(\det L)^{1/m}$. If $s \in \mathbf{Z}$, $|s| \le H$, and $\gcd\{u-s,n\} \ge \lambda$, then $\varphi(s/H) = 0$.*

Compare to the case $v = 1$, $w = 1$, $d = 1$ of Theorem 5.1.

Proof. Define $r = s/H$. By hypothesis $g_1 = H \ge 1$; $1/f_d = n/H \ge 1$; $\alpha = \sqrt{1 + \lg(1/f_d)/\lg(2g_1)}$; $r \in \mathbf{Q}$; $|r| = |s|/H \le 1$; $g(r) = s \in \mathbf{Z}$; and $f(r) = (s-u)/n$, so $\gcd\{1, f(r)\} \ge \lambda/n = m^{1/2k}(2g_1)^{\alpha(1+1/2k)} f_d/g_1$. Apply Theorem 3.1. □

Theorem 7.2. *Let n, u, H be positive integers such that $n \ge H$. Define $\alpha = \sqrt{(\lg 2n)/\lg 2H}$, $k = \lceil \alpha \lceil \lg 2H \rceil /2 \rceil$, $m = \lceil \alpha(k+1) \rceil$, $f = (Hx-u)/n \in \mathbf{Q}[x]$, $g = Hx \in \mathbf{Q}[x]$, $d = 1$, and L as above. Let $\varphi \in L$ be a nonzero vector such that $|\varphi| \le 2^{(m-1)/2}(\det L)^{1/m}$. If $s \in \mathbf{Z}$, $|s| \le H$, and $\gcd\{u-s,n\} \ge 2m^{1/2k}(2H)^{\alpha}$, then $\varphi(s/H) = 0$.*

Proof. By hypothesis $2H \ge 2$ so $\lg 2H \ge 1$; hence k is a positive integer. Also $2k \ge \alpha \lg 2H$ so $2 \ge (2H)^{\alpha/2k}$ so $\gcd\{u-s,n\} \ge \lambda$ where $\lambda = m^{1/2k}(2H)^{\alpha(1+1/2k)}$. Apply Theorem 7.1. □

Theorem 7.3. *Let n, u, H be positive integers such that $n \ge H$. Let k be a positive integer. Define $\alpha = \sqrt{(\lg 2n)/\lg 2H}$ and $m = \lceil \alpha(k+1) \rceil$. Then there are at most $m-1$ integers $s \in \{-H, \ldots, 0, 1, \ldots, H\}$ such that $\gcd\{u-s,n\} \ge m^{1/2k}(2H)^{\alpha(1+1/2k)}$.*

Proof. Apply lattice-basis reduction to Theorem 7.1. □

Theorem 7.4. *Let n, u, H be positive integers such that $n \ge H$. Define $\alpha = \sqrt{(\lg 2n)/\lg 2H}$, $k = \lceil \alpha \lceil \lg 2H \rceil /2 \rceil$, and $m = \lceil \alpha(k+1) \rceil$. Then there are at most $m-1$ integers $s \in \{-H, \ldots, 0, 1, \ldots, H\}$ with $\gcd\{u-s,n\} \ge 2m^{1/2k}(2H)^{\alpha}$.*

Proof. Apply lattice-basis reduction to Theorem 7.2. □

History. The rational-function-field version of the simple case $k = 1$, $m = 2$ is the "Berlekamp-Massey algorithm" for decoding "Reed-Solomon codes."

The fact that one can efficiently correct larger errors was pointed out in the function-field case by Sudan in [33], and in the number-field case by Goldreich, Ron, and Sudan in [14]. These results are tantamount to optimizing m in Theorem 2.3 with $k = 1$. Increasing k produces an asymptotic $\sqrt{2}$ exponent improvement, as discussed in Section 3; this $\sqrt{2}$ improvement was pointed out in the function-field case by Guruswami and Sudan in [17], and in the number-field case by Boneh in [5].

Algorithms that may produce several values of s are often called "list decoding" algorithms. Of course, the resulting list is most useful when it has just one value of s.

A numerical example. Define $H = 1000000$, $n = 101 \cdot 103 \cdot 107 \cdot 109 \cdot 113 \cdot 127 \cdot 131 \cdot 137 \cdot 139 \cdot 149 \cdot 151 \cdot 157 \cdot 163 \cdot 167 \cdot 173 \cdot 179 \cdot 181 \cdot 191 \cdot 193 \cdot 197 \cdot 199$, and $u = 476534584519360044215357448296811494656848207$. The goal here is to find every $s \in [-H, H]$ with residue representation close to $(u \bmod 101, \ldots, u \bmod 199) = (94, 43, 17, 71, 103, 77, 64, 25, 114, 9, 106, 16, 62, 134, 75, 13, 155, 26, 138, 21, 105)$.

Choose $k = 3$. Define $\alpha = \sqrt{(\lg 2n)/\lg 2H} \approx 2.697$ and $m = \lceil \alpha(k+1) \rceil = 11$. Define $f = (Hx - u)/n$, $g = Hx$, and $L = \mathbf{Z} + \mathbf{Z}f + \mathbf{Z}f^2 + \mathbf{Z}f^3 + \mathbf{Z}gf^3 + \mathbf{Z}g^2f^3 + \mathbf{Z}g^3f^3 + \mathbf{Z}g^4f^3 + \mathbf{Z}g^5f^3 + \mathbf{Z}g^6f^3 + \mathbf{Z}g^7f^3$.

Reduce the lattice basis $1, f, f^2, f^3, gf^3, g^2f^3, g^3f^3, g^4f^3, g^5f^3, g^6f^3, g^7f^3$ to find a nonzero vector in L of length at most $2^{(m-1)/2}(\det L)^{1/m}$: for example, the vector

$$\begin{aligned}
&(25587000000000000000000000000000000000000000000000000/n^3)x^8\\
&-(11721491973000000000000000000000000000000000000000000/n^3)x^7\\
&-(16962979598492916000000000000000000000000000000000000/n^3)x^6\\
&-(9080495056404508812155000000000000000000000000000000/n^3)x^5\\
&-(16268872584260106363071226779000000000000000000000000/n^3)x^4\\
&-(4786092732625488403021583597543361000000000000000000/n^3)x^3\\
&-(68525665600669610580610714527465995863869000000000000/n^3)x^2\\
&-(4866470374300829151096400546244449180155160401000000/n^3)x\\
&+(19654220351564720341671319570621613333314080770830407/n^3)1.
\end{aligned}$$

The only rational root of this polynomial is s/H where $s = 476511$. The vector $(94, 33, 40, 72, 103, 7, 64, 25, 19, 9, 106, 16, 62, 60, 69, 13, 119, 157, 187, 165, 105)$ is the residue representation of s; the distance from s to u is approximately 79.41.

Theorem 7.1 guaranteed that this procedure would find every s within distance $\lg n - \lg \lambda \approx 84.8$ of u; here $\lambda = m^{1/2k}(2H)^{\alpha(1+1/2k)}$. Even better, Theorem 2.3 guaranteed that this procedure would find every s within distance $-\lg \gamma \approx 88.28$ of u; here $\gamma = m^{1/2k}(2H)^{(m-1)/2k}n^{(k+1)/2m-1}$. Both bounds are far above $(\lg n - \lg 2H)/2 \approx 65.16$.

References

[1] —, *Annual ACM symposium on theory of computing: proceedings of the 31st symposium (STOC '99) held in Atlanta, GA, May 1–4, 1999*, Association for Computing Machinery, New York, 1999. ISBN 1–58113–067–8. MR 2001f:68004. See [14].

[2] —, *Proceedings of the 32nd annual ACM symposium on theory of computing*, Association for Computing Machinery, New York, 2000. ISBN 1–58113–184–4. See [5].

[3] Leonard M. Adleman, Carl Pomerance, Robert S. Rumely, *On distinguishing prime numbers from composite numbers*, Annals of Mathematics **117** (1983), 173–206. ISSN 0003–486X. MR 84e:10008. Citations in this paper: §5.

[4] Daniel Bleichenbacher, *Compressing Rabin signatures*, in [29] (2004), 126–128. Citations in this paper: §4.

[5] Dan Boneh, *Finding smooth integers in short intervals using CRT decoding*, in [2] (2000), 265–272; see also newer version [6]. Citations in this paper: §1, §1, §1, §7.

[6] Dan Boneh, *Finding smooth integers in short intervals using CRT decoding*, Journal of Computer and System Sciences **64** (2002), 768–784; see also older version [5]. ISSN 0022–0000. MR 1 912 302. URL: `http://crypto.stanford.edu/~dabo/abstracts/CRTdecode.html`.

[7] Dan Boneh, Glenn Durfee, Nick Howgrave-Graham, *Factoring $N = p^r q$ for large r*, in [35] (1999), 326–337. URL: `http://crypto.stanford.edu/~dabo/abstracts/prq.html`. Citations in this paper: §1, §1, §5.

[8] Don Coppersmith, *Finding a small root of a univariate modular equation*, in [27] (1996), 155–165; see also newer version [10]. MR 97h:94008. Citations in this paper: §1, §1, §1, §3.

[9] Don Coppersmith, *Finding a small root of a bivariate integer equation; factoring with high bits known*, in [27] (1996), 178–189; see also newer version [10]. MR 97h:94009. Citations in this paper: §1, §5.

[10] Don Coppersmith, *Small solutions to polynomial equations, and low exponent RSA vulnerabilities*, Journal of Cryptology **10** (1997), 233–260; see also older version [8] and [9]. ISSN 0933–2790. MR 99b:94027.

[11] Don Coppersmith, *Finding small solutions to small degree polynomials*, in [32] (2001), 20–31. MR 2003f:11034. URL: `http://cr.yp.to/bib/entries.html#2001/coppersmith`. Citations in this paper: §3, §3.

[12] Don Coppersmith, Nick Howgrave-Graham, S. V. Nagaraj, *Divisors in residue classes, constructively* (2004). URL: `http://eprint.iacr.org/2004/339`. Citations in this paper: §5, §6.

[13] Michael Darnell (editor), *Cryptography and coding: proceedings of the 6th IMA International Conference held at the Royal Agricultural College, Cirencester, December 17–19, 1997*, Lecture Notes in Computer Science, 1355, Springer-Verlag, 1997. ISBN 3–540–63927–6. MR 99g:94019. See [19].

[14] Oded Goldreich, Dana Ron, Madhu Sudan, *Chinese remaindering with errors*, in [1] (1999), 225–234; see also newer version [15]. MR 2001i:68050. URL: `http://theory.lcs.mit.edu/~madhu/papers.html`. Citations in this paper: §1, §1, §7.

[15] Oded Goldreich, Dana Ron, Madhu Sudan, *Chinese remaindering with errors*, IEEE Transactions on Information Theory **46** (2000), 1330–1338; see also older version [14]. ISSN 0018–9448. MR 2001k:11005. URL: `http://theory.lcs.mit.edu/~madhu/papers.html`.

[16] Ronald L. Graham, Jaroslav Nešetřil (editors), *The mathematics of Paul Erdős. I*, Algorithms and Combinatorics, 13, Springer-Verlag, Berlin, 1997. ISBN 3–540–61032–4. MR 97f:00032. See [22].

[17] Venkatesan Guruswami, Madhu Sudan, *Improved decoding of Reed-Solomon and algebraic-geometry codes*, IEEE Transactions on Information Theory **45** (1999), 1757–1767. ISSN 0018–9448. MR 2000j:94033. URL: `http://theory.lcs.mit.edu/~madhu/bib.html`. Citations in this paper: §1, §7.

[18] Johan Håstad, *Solving simultaneous modular equations of low degree*, SIAM Journal on Computing **17** (1988), 336–341. ISSN 0097–5397. MR 89e:68049. URL: `http://www.nada.kth.se/~johanh/papers.html`. Citations in this paper: §1.

[19] Nicholas Howgrave-Graham, *Finding small roots of univariate modular equations revisited*, in [13] (1997), 131–142. MR 99j:94049. Citations in this paper: §1, §1, §1, §5.

[20] Nicholas Howgrave-Graham, *Computational mathematics inspired by RSA*, Ph.D. thesis, 1998. URL: `http://cr.yp.to/bib/entries.html#1998/howgrave-graham`. Citations in this paper: §1, §3, §5, §6.

[21] Nicholas Howgrave-Graham, *Approximate integer common divisors*, in [32] (2001), 51–66. MR 2003h:11160. URL: `http://cr.yp.to/bib/entries.html#2001/howgrave-graham`. Citations in this paper: §1.

[22] Sergei Konyagin, Carl Pomerance, *On primes recognizable in deterministic polynomial time*, in [16] (1997), 176–198. MR 98a:11184. URL: `http://cr.yp.to/bib/entries.html#1997/konyagin`. Citations in this paper: §1, §5.

[23] Arjen K. Lenstra, Hendrik W. Lenstra, Jr., László Lovász, *Factoring polynomials with rational coefficients*, Mathematische Annalen **261** (1982), 515–534. ISSN 0025–5831. MR 84a:12002. URL: `http://cr.yp.to/bib/entries.html#1982/lenstra-lll`. Citations in this paper: §2.

[24] Hendrik W. Lenstra, Jr., *Divisors in residue classes*, Mathematics of Computation **42** (1984), 331–340. ISSN 0025–5718. MR 85b:11118. URL: `http://www.jstor.org/sici?sici=0025-5718(198401)42:165<331:DIRC>2.0.CO;2-6`. Citations in this paper: §1, §1, §5.

[25] Hendrik W. Lenstra, Jr., *Lattices* (2006); chapter in this book. Citations in this paper: §2.

[26] Rüdiger Loos, *Computing rational zeros of integral polynomials by p-adic expansion*, SIAM Journal on Computing **12** (1983), 286–293. ISSN 0097–5397. MR 85b:11123. Citations in this paper: §2.

[27] Ueli M. Maurer (editor), *Advances in cryptology—EUROCRYPT '96: Proceedings of the Fifteenth International Conference on the Theory and Application of Cryptographic Techniques held in Saragossa, May 12–16, 1996*, Lecture Notes in Computer Science, 1070, Springer-Verlag, Berlin, 1996. ISBN 3–540–61186–X. MR 97g:94002. See [8], [9].

[28] Teo Mora (editor), *Applied algebra, algebraic algorithms and error-correcting codes: proceedings of the sixth international conference (AAECC-6) held in Rome, July 4–8, 1988*, Lecture Notes in Computer Science, 357, Springer-Verlag, Berlin, 1989. ISBN 3–540–51083–4. MR 90d:94002. See [34].

[29] Tatsuaki Okamoto (editor), *Topics in cryptology—CT-RSA 2004: the cryptographers' track at the RSA Conference 2004, San Francisco, CA, USA, February 23–27, 2004, proceedings*, Lecture Notes in Computer Science, Springer, Berlin, 2004. ISBN 3–540–20996–4. MR 2005d:94157. See [4].

[30] Franz Pichler (editor), *Advances in cryptology—EUROCRYPT '85: proceedings of a workshop on the theory and application of cryptographic techniques (EUROCRYPT '85) held in Linz, April 1985*, Lectures Notes in Computer Science, 219, Springer-Verlag, 1986. ISBN 3–540–16468–5. MR 87d:94003. See [31].

[31] Ronald L. Rivest, Adi Shamir, *Efficient factoring based on partial information*, in [30] (1986), 31–34. MR 851 581. Citations in this paper: §1, §5.

[32] Joseph H. Silverman (editor), *Cryptography and lattices: proceedings of the 1st International Conference (CaLC 2001) held in Providence, RI, March 29–30, 2001*, Lecture Notes in Computer Science, 2146, Springer-Verlag, Berlin, 2001. ISBN 3–540–42488–1. MR 2002m:11002. See [11], [21].

[33] Madhu Sudan, *Decoding of Reed Solomon codes beyond the error-correction bound*, Journal of Complexity **13** (1997), 180–193. ISSN 0885–064X. MR 98f:94024. URL: `http://theory.lcs.mit.edu/~madhu/bib.html`. Citations in this paper: §1, §7.

[34] Brigitte Vallée, Marc Girault, Philippe Toffin, *How to guess ℓth roots modulo n by reducing lattice bases*, in [28] (1989), 427–442. MR 90k:11168. URL: `http://cr.yp.to/bib/entries.html#1989/vallee`. Citations in this paper: §1.

[35] Michael Wiener (editor), *Advances in cryptology—CRYPTO '99*, Lecture Notes in Computer Science, 1666, Springer-Verlag, Berlin, 1999. ISBN 3–5540–66347–9. MR 2000h:94003. See [7].

Department of Mathematics, Statistics, and Computer Science, The University of Illinois at Chicago, Chicago, IL 60607–7045

E-mail address: `djb@cr.yp.to`