Conceptual Aspects to solve Smale's 17–th Problem: complexity, probability, polynomial equations and Integral Geometry. *

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Smale's List

18 Problems, as....

Problem 1: The Riemann Hypothesis

Problem 2: The Poincaré Conjecture (Perelman)

Problem 3: Does P = NP ?

Problem 4: Integer Zeros of a Polynomial.

Problem 5: Height Bounds for diophantine curves.

$\cdots$

Problem 9: The Linear Programming Problem.

$\cdots$

Problem 14: The Lorentz Attractor Problem. (Tucker, 02)

$\boxed{\textbf{17-th Problem.}}$

Can a zero of n complex polynomial equations in n unknowns be found approximately on the average, in polynomial time with a uniform algorithm?.

(Beltrán-**P.** , 06)

Historical Scketch

XIX-th century: Modern Elimination Theory
Bézout, Cayley, Hilbert, Kronecker, Sturm, Sylvester

1900–1930: *Macaulay, König,...*

1930–1965: Vanished on the air?

1965–: Monomial orders and standard–Gröbner Basis *Hironaka, Buchberger,...,Rewriting Techniques*

Sparse Approach... *Bernstein, Kouchnirenko, Sturmfels....*

Complexity Classes Approach... *Cook [* $\mathbf{P} = \mathbf{NP}$ *?]*

1995–: Intrinsical Methods adapted to data structures
TERA, KRONECKER ....

A Simplified version

Goal: *Efficient Algorithmics for Problems Given by Polynomial Equations*

Potential Applications : † Information Theory (Coding, Crypto,...), Game Theory, Graphic and Mechanical Design, Chemist, Robotics, ...

The Problem: Efficiency

Rk. *Most algorithms for Elimination Problems run in worse than exponential time in the number of variables:*

Intractable for Practical Applications.

†Many of them Casual but not Causal

SOLVING

INPUT: A list of multivariate polynomial equations: $f_1, \ldots, f_s \in \mathbb{C}[X_1, \ldots, X_n]$.

OUTPUT: A description of the solution variety
$V(f_1, \ldots, f_s) := \{x \in \mathbb{C}^n \ : \ f_i(x) = 0 ...\}$.

Description: The kind of description determines the kind of problems/questions you may answer about $V(f_1, \ldots, f_s)$

Example: Symbolic/Algebraic Computing $\longrightarrow$ questions involving quantifiers

Hilbert's Nulltellensatz (HN) *Given $f_1, \ldots, f_s$ decide whether the following holds:*

$$\exists x \in \mathbb{C}^n \ \ f_i(x) = 0, \ \ 1 \leq i \leq s.$$

DIFFERENT SCHOOLS

Syntactic Standard, Gröbner Basis, Rewriting...a Long List

Structural :Find the suitable complexity class for the problem NP-hard, PSPACE,...

Semi–Semantics: Using combinatorial objects (hence semi-semantic) to control complexity: Sparse School: using Newton polytopes Bernstein. Kouchnirenko, Sturmfels...

Semantic/Intrinsic: Mostly the TERA group: *Cantabria (P., Morais, Montaña, Hägele,...); Polytechnique (Giusti, Bostan, Lecerf, Schost, Salvy...); * Buenos Aires (Heintz, Krick, Matera, Solerno, ...); * Humboldt (Bank, Mbakop,Lehmann)*

- Polynomials viewed as programs.

- Parameters of Semantical Objects (algebraic varieties) dominate complexity.

Degree of V *([Heintz,83], [Vogel, 83], [Fulton, 81]) :♯ of intersection points with generic linear varieties.*

Height of V:
Bit length of the coefficients CHOW FORM

* *Geometric Degree of a Sequence:*

$$\delta(V_1, \ldots, V_r) := max\{deg(V_i) \ : \ 1 \leq i \leq r\}.$$

Theorem 1 *There is a bounded error probability Turing machine that answers* **HN** *in time polynomial in*

$$L\ \delta\ H,$$

where

L *is the input length (whatever usual data structure),*

δ *is the geometric degree of a deformation sequence (Kronecker's deformation) and*

H *is the height of the last equi-dimensional variety computed.*

Examples

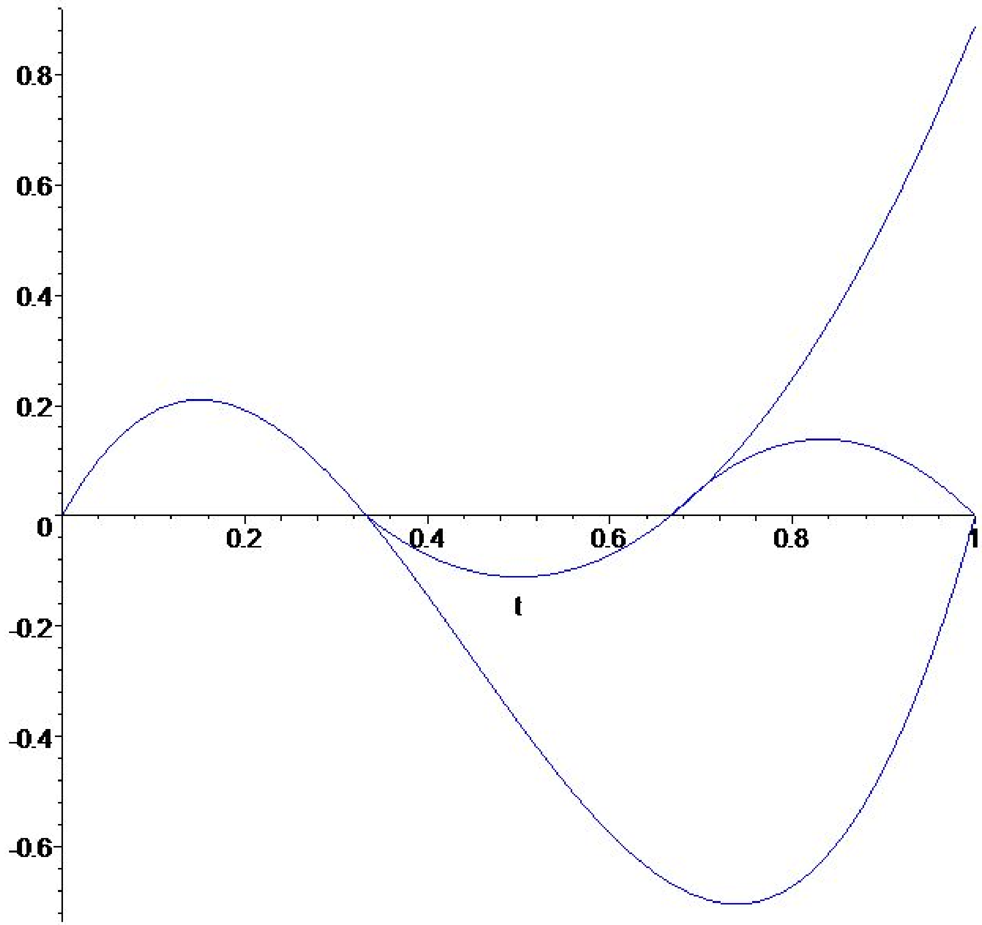
$$X_1^2 - X_1 = 0, \ldots, X_n^2 - X_n = 0, k - \sum_{i=1}^{n} m_i X_i = 0.$$

$$X_1^2 - X_1 = 0, \ldots, X_n^2 - X_n = 0, k - \sum_{i=1}^{n} 2^{i-1} X_i = 0.$$

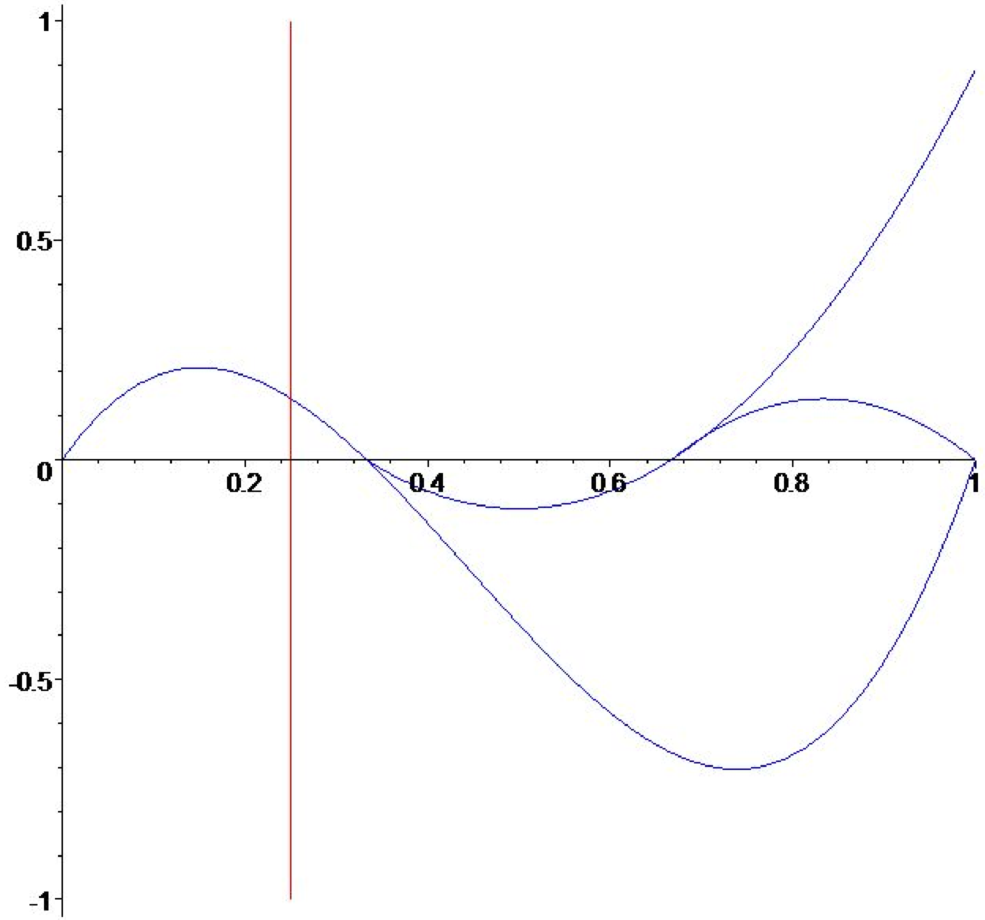
$$X_1^2 - X_1 = 0, \ldots, X_n^2 - X_n = 0, 512 - \sum_{i=1}^{n} 2^{i-1} X_i = 0.$$

$$X_2^2 - X_1 = 0, X_3^2 - X_2 = 0 \ldots, X_n^2 - X_{n-1} = 0, k - X_n = 0.$$

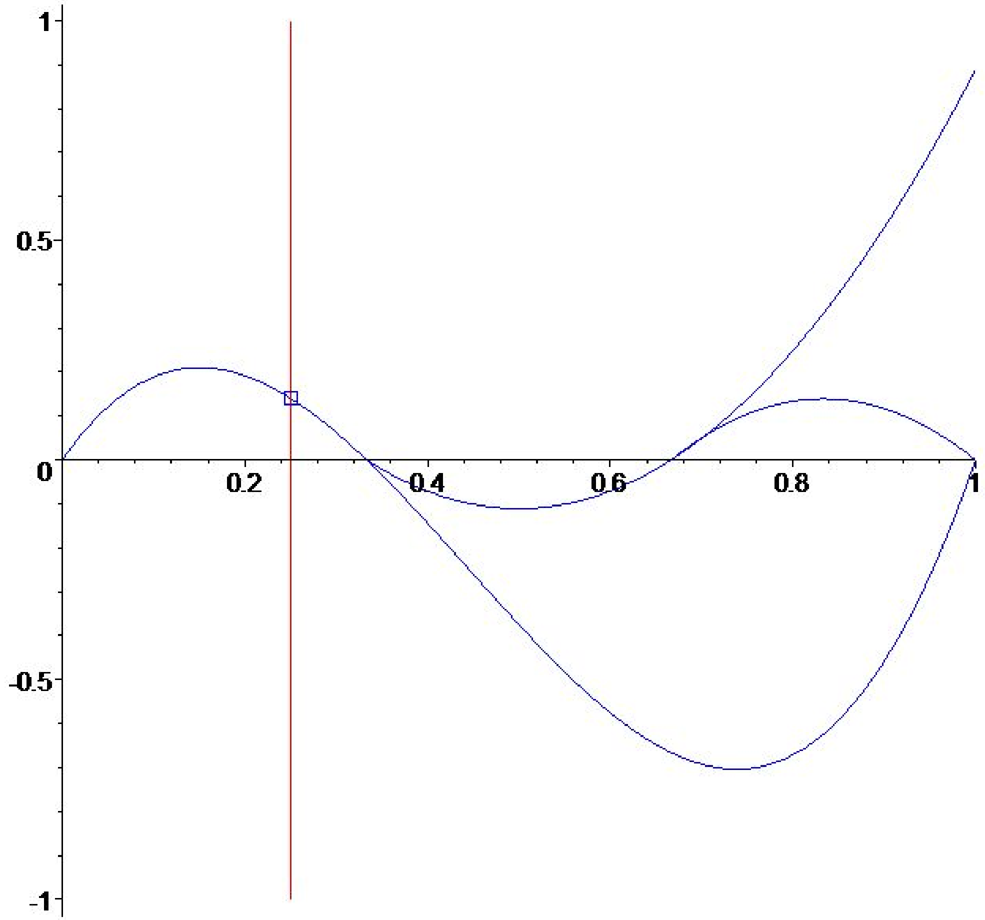
Kronecker's Deformation



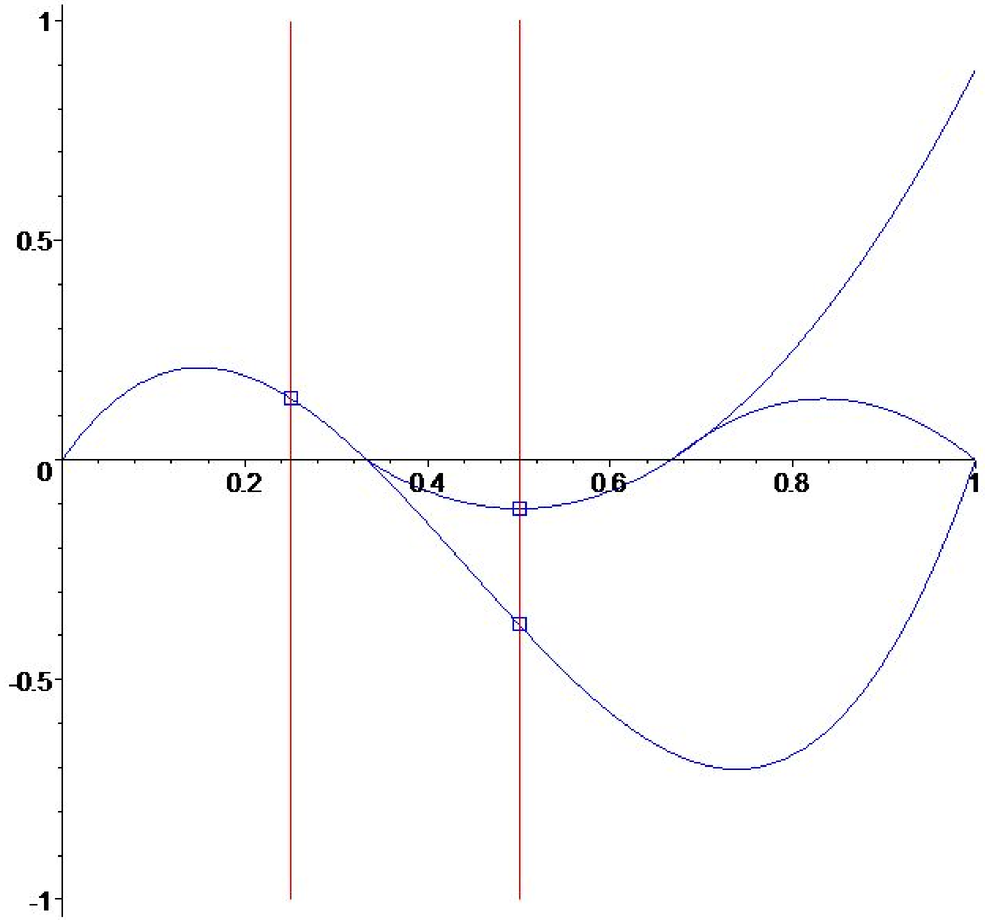
Initialize



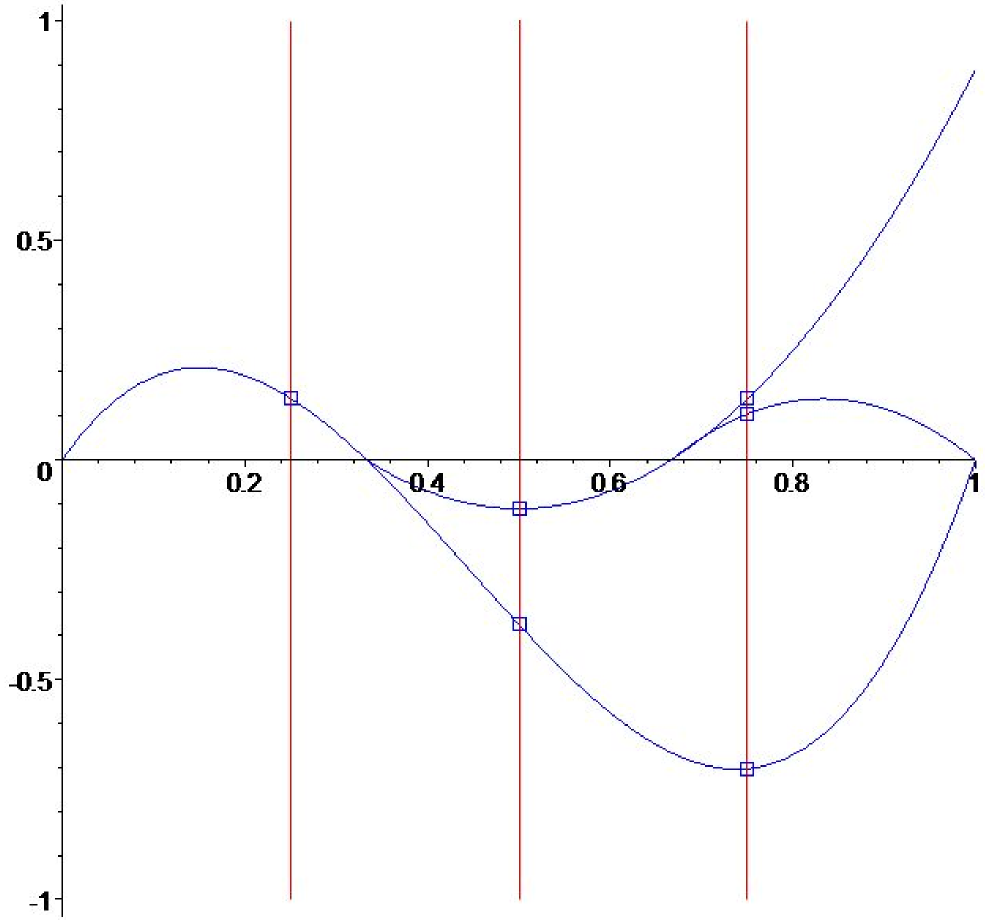
Lifting Fibers



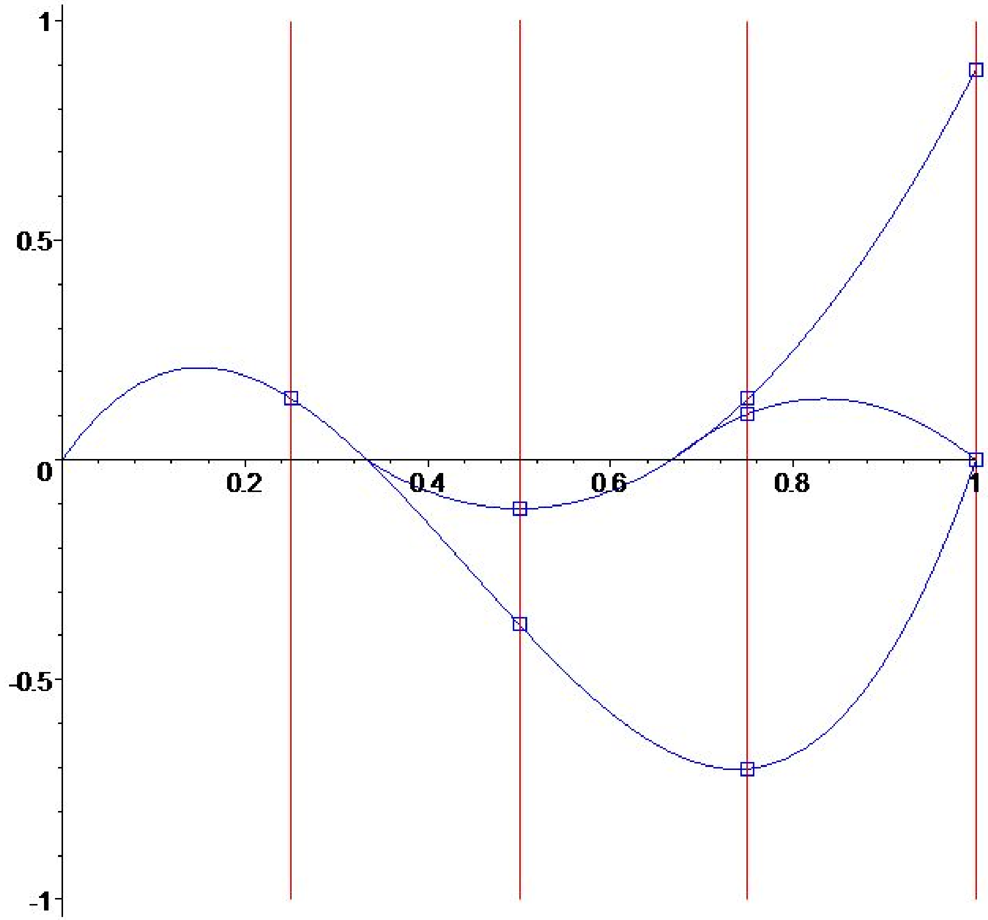
Jumping from a Lifting Fiber to a new one



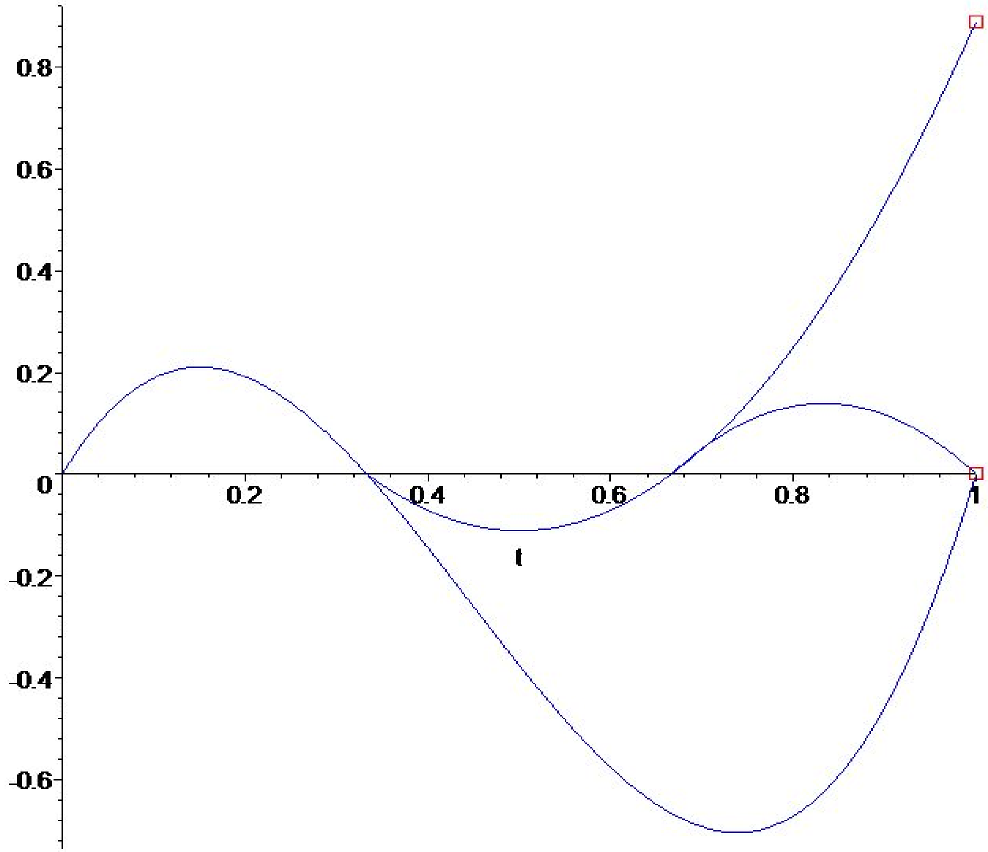
And so on...



Until the End



The target



We got:

A description of the target variety through a birational isomorphism, even biregular in the zero–dimensional case, that contains information that suffices to answer elimination questions.

But...

Is that optimal in terms of complexity ?

Algorithms based on **a deformation***:*
A sequence suite $V_1, \ldots, V_n$ of intermediate varieties to solve before "eliminating"

Universal Solving
An algorithm is called **Universal** *if its output contains information enough about the variety of solutions to answer* **all** *elimination questions.*

Remark 2 *Most Computer Algebra/Symbolic Computation procedures are Universal.*

Lower Complexity Bounds

Theorem [Castro-Giusti-Heintz-Matera-P.,2003]
Any universal solving procedure requires exponential running time.

* *TERA algorithm is* **essentially optimal.**

* *Running time is greater than the* **Bézout Number:**

$$\prod_{i=1}^{n} deg(f_i) \geq 2^n.$$

* *No Universal solving procedure may improve this lower complexity bound.*

Searching Non–Universal Solving Procedures.

Searching for procedures that compute partial (non–universal) information about the solution variety in polynomial running time.

Smale's 17th Problem

What is "Partial Information"?

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For instance, a "good approximation" to some of the solutions

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Example

INPUT: $f_1, \ldots, f_n \in \mathbb{Q}[X_1, \ldots, X_n]$ t.q. $\sharp V(f_1, \ldots, f_n) < \infty$.

OUPUT: $z \in \mathbb{Q}[i]^n$ such that there exists $\zeta \in V(f_1, \ldots, f_n)$ satisfying

$$||\zeta - z|| < \varepsilon,$$

for som $\varepsilon > 0$.

Some Multivariate Elimination and some lattice reduction algorithms (under KLL approach) yield

Theorem 3 (Castro-Hagele-Morais-P., 01) *There is a computational equivalence between:*

- *Approimations* $z \in \mathbb{Q}[i]^n$ *of some of the zeros*$\zeta \in V(f_1, \ldots, f_n)$,
- *A description "á la Kronecker–TERA" of the residual class field of* $\mathbb{Q}_\zeta$.

Theorem (cont.)

The running time of this computational equivalence is polynomial in:

- $D_\zeta =$ *degree of the residual class field* $\mathbb{Q}_\zeta$.
- $L =$ *input size.*
- $H_\zeta =$ *height of the residual class field* $\mathbb{Q}_\zeta$.

Namely, **a "good" approximation contains information that suffices for elimination**(although it is not clear whether you should compute it)

Immediate Application

Theorem 4 *There is an algorithm that performs the following taks:*

- Input: *A univariate polynomial $f \in \mathbb{Q}[T]$.*

- Output: *A primitive element description of the normal closure of f.*

THe running time of this procedure is polynomial in the following quatities:

$$d, h, \sharp Gal_{\mathbb{Q}}(f),$$

where d is the degree of f and h is the bit length of the coefficients of f.

Remark: A geometric algorithm such that the complexity is not of order $d!$ except when unavoidable.

Good Approximation?

For simplicity we work on the projective space

Systems of homogeneous polynomials:

$$F := [f_1, \ldots, f_n] \in \mathcal{H}_{(d)},$$

$deg(f_i) = d_i$, $(d) := (d_1, \ldots, d_n)$,
$\mathcal{H}_{(d)} :=$ Complex vector space of all equations of given degree.

$$V_{\mathbb{P}}(F) := \{x \in \mathbb{P}_n(\mathbb{C}) \ : \ F(x) = 0\}.$$

The incidence variety (Room-Kempf, Shub-Smale)

$$V := \{(F, x) \in \mathbb{P}(\mathcal{H}_{(d)} \times \mathbb{P}_n(\mathbb{C}) \ : \ F(x) = 0\}.$$

PROJECTIVE NEWTON'S OPERATOR

(M. Shub amd S. Smale 1986–1996)

$$\pi \ : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}_n(\mathbb{C})$$

Notations: *Projective Metrics :*

Riemannian :

$$d_R(\pi(x), \pi(x')) := arccos \left(\frac{|\langle x, x' \rangle|}{\|x\| \|x'\|} \right).$$

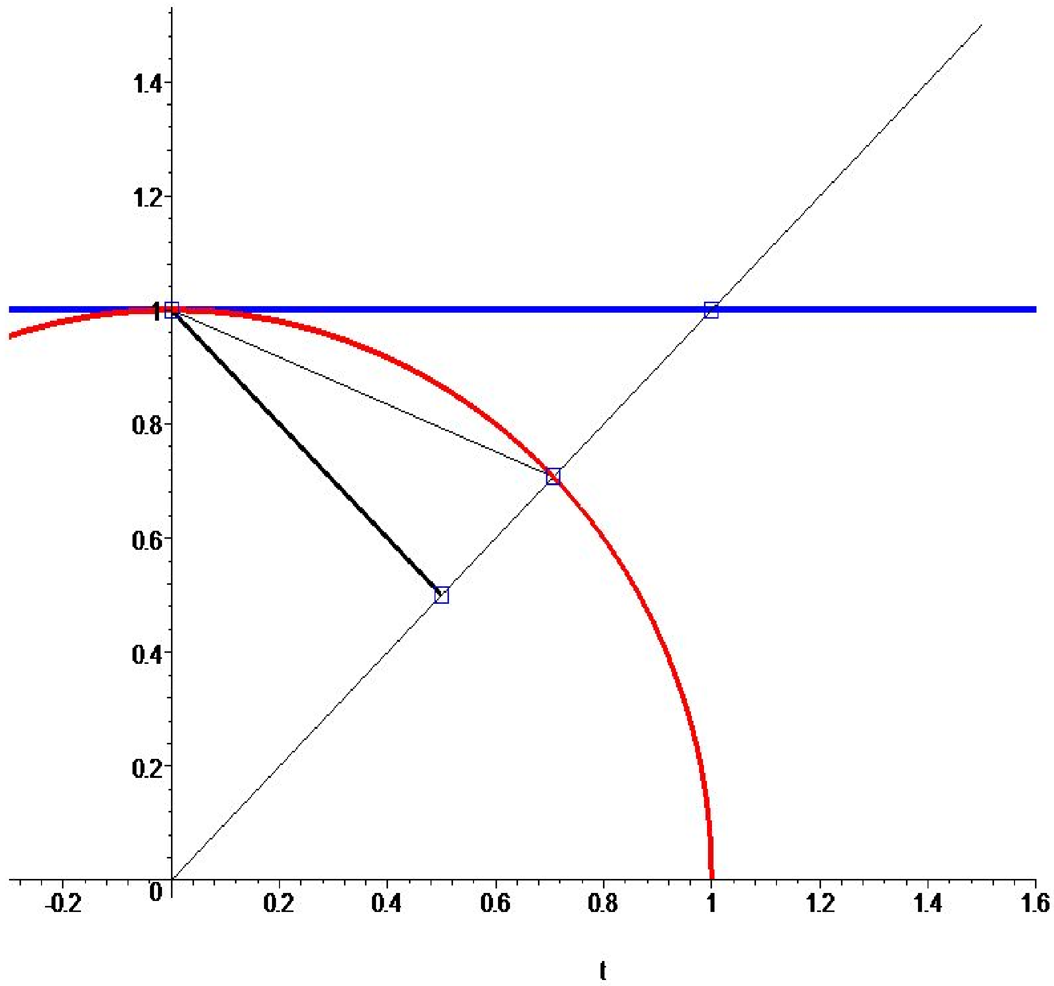
Fubini–Study :

$$d_P(\pi(x), \pi(x')) := sin\ d_R(\pi(x), \pi(x')).$$

Tangent Distance :

$$d_T(\pi(x), \pi(x')) := tan d_R(\pi(x), \pi(x')).$$

A Picture



Newton's Operator II

Tangent Space at a point $z \in \mathbb{P}_n(\mathbb{C})$:

$$T_z\mathbb{P}_n(\mathbb{C}) := \{w \in \mathbb{C}^{n+1} \ : \ \langle w, z\rangle = 0\}.$$

A system of polynomial equations $F := [f_1, \ldots, f_n]$, Jacobian matrix :

$$DF(z) \ : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^n.$$

If z is not a critical point, the restriction to the tangent space:

$$T_z f := DF(z) \mid_{T_z} \ : T_z\mathbb{P}_n(\mathbb{C}) \longrightarrow \mathbb{C}^n.$$

The inverse:

$$(T_z f)^{-1} \ : \mathbb{C}^n \longrightarrow \mathbb{C}^{n+1}.$$

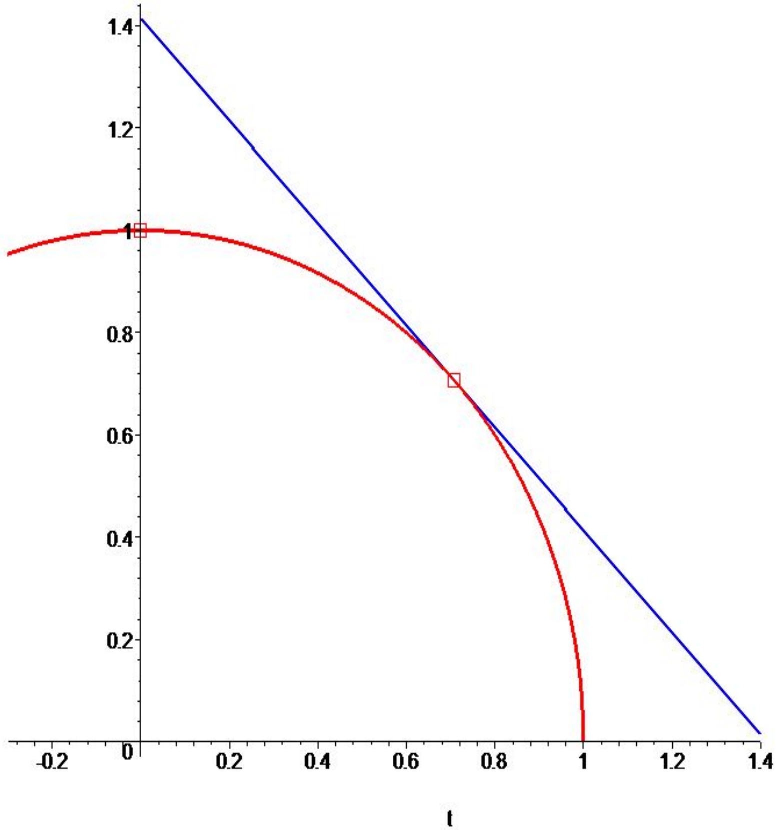
Projective Newton's Operator III

The canonical projection $\pi \ : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}_n(\mathbb{C})$.

For every non-critical $\pi(z) \in \mathbb{P}_n(\mathbb{C})$ Newton's operator is given by:

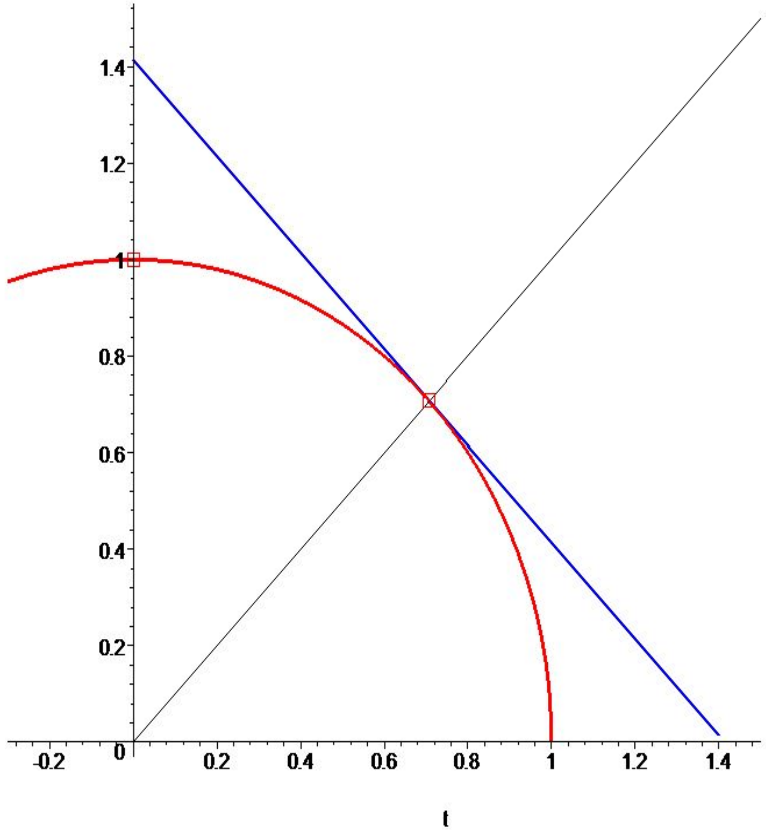
$$N_F(\pi(z)) := \pi\left(z - \left(DF(z)\mid_{T_z}\right)^{-1} F(z)\right),$$

Some pictures I



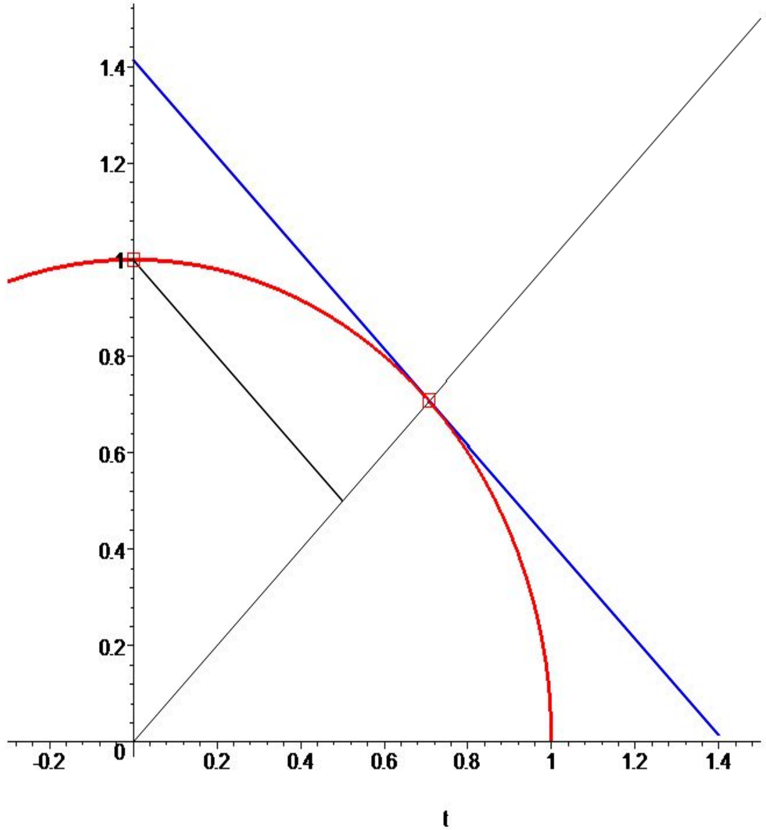
$$T_z\mathbb{P}_n(\mathbb{C})$$

N. Op. Picture II



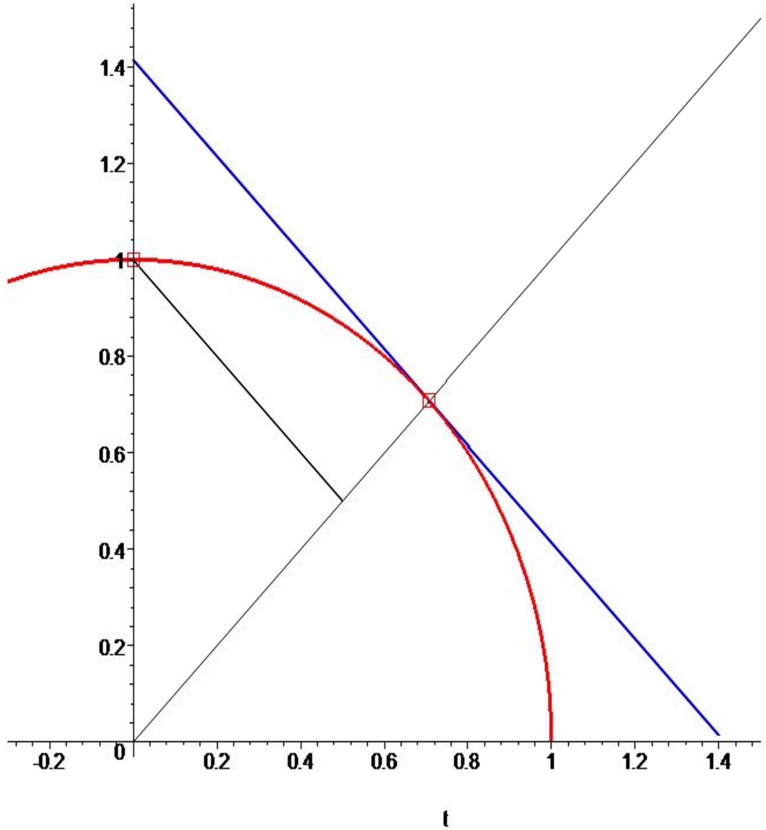
$$T_z\mathbb{P}_n(\mathbb{C})$$

N. Op. Picture III



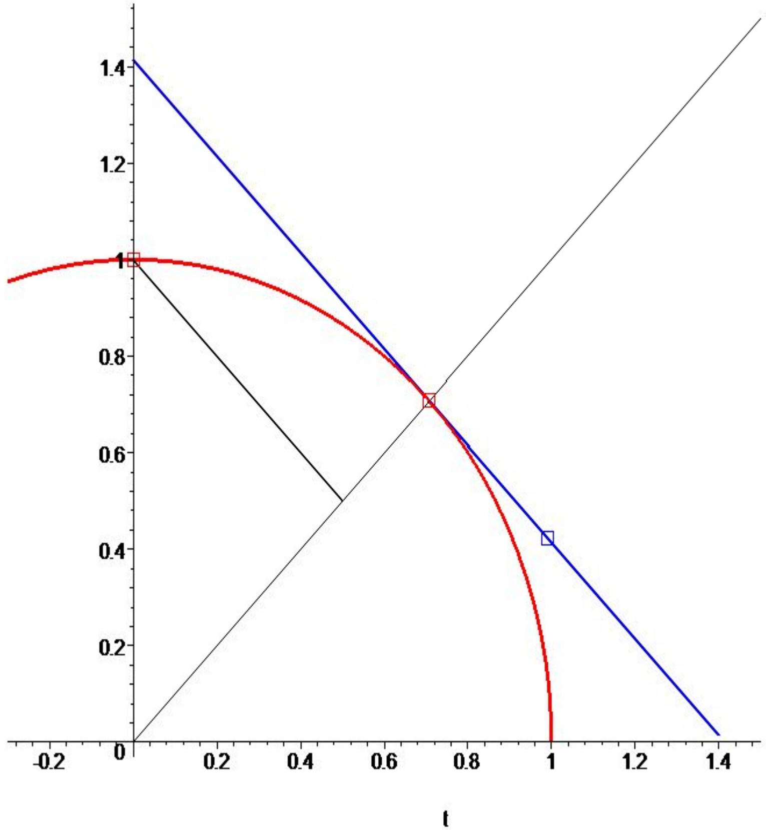
$T_z\mathbb{P}_n(\mathbb{C})$

N. Op. Picture IV



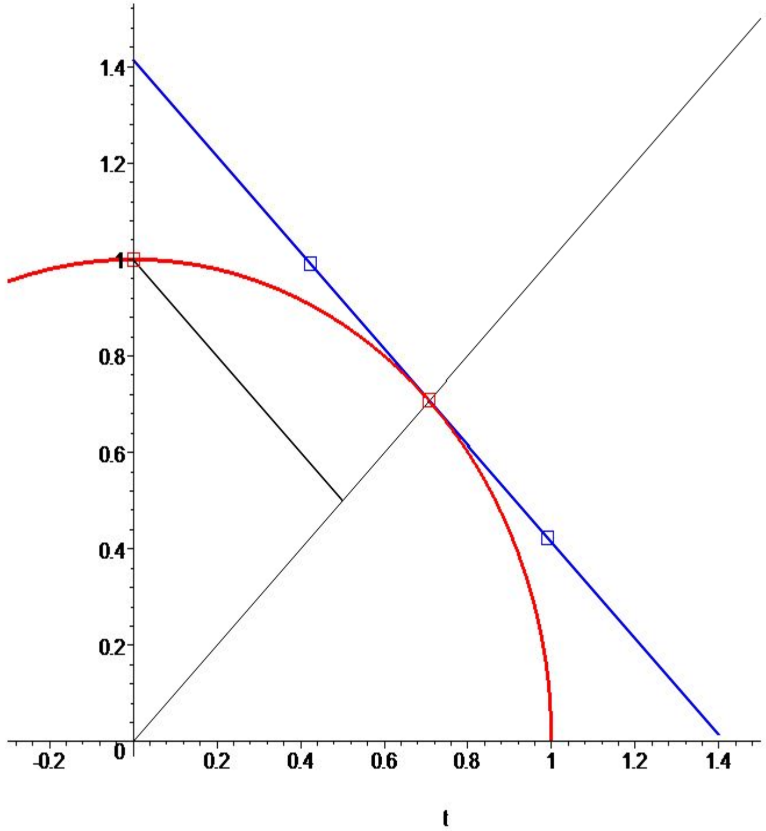
$f(z) \in T_0\mathbb{C}^n = \mathbb{C}^n \qquad T_z\mathbb{P}_n(\mathbb{C})$

$\boxed{\text{N. Op. Picture V}}$



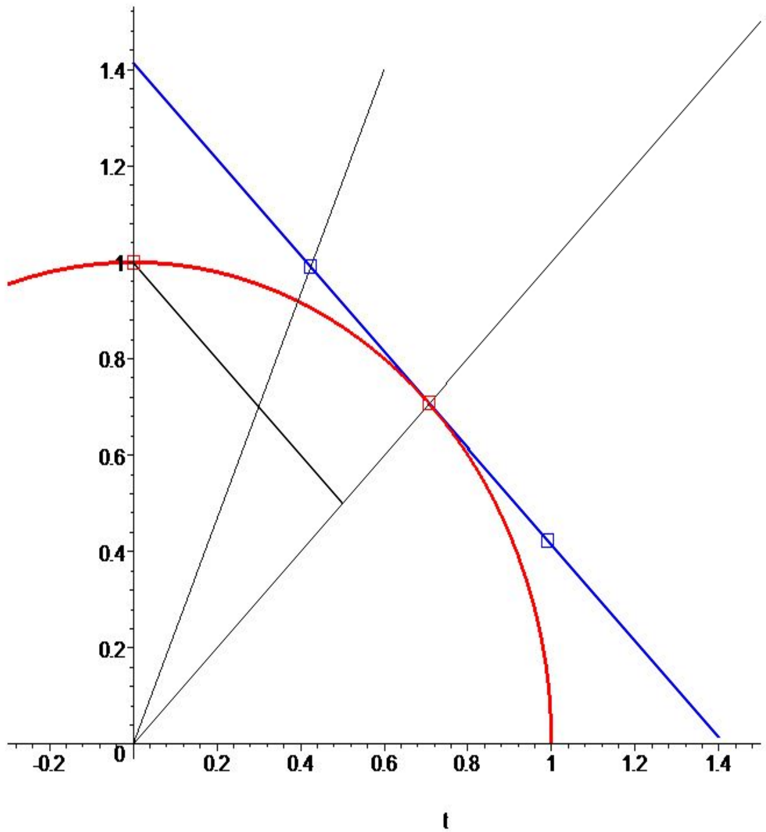
$$T_z f^{-1} f(z) \in T_z \mathbb{P}_n(\mathbb{C})$$

N. Op. Picture VI



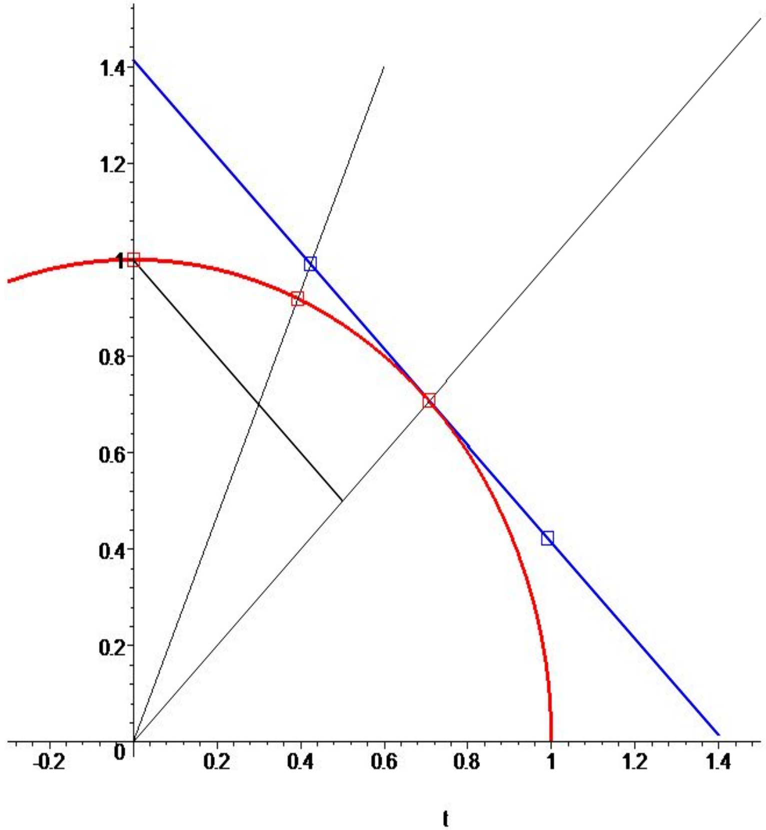
$$-T_z f^{-1} f(z) \in T_z \mathbb{P}_n(\mathbb{C})$$

N. Op. Picture VII



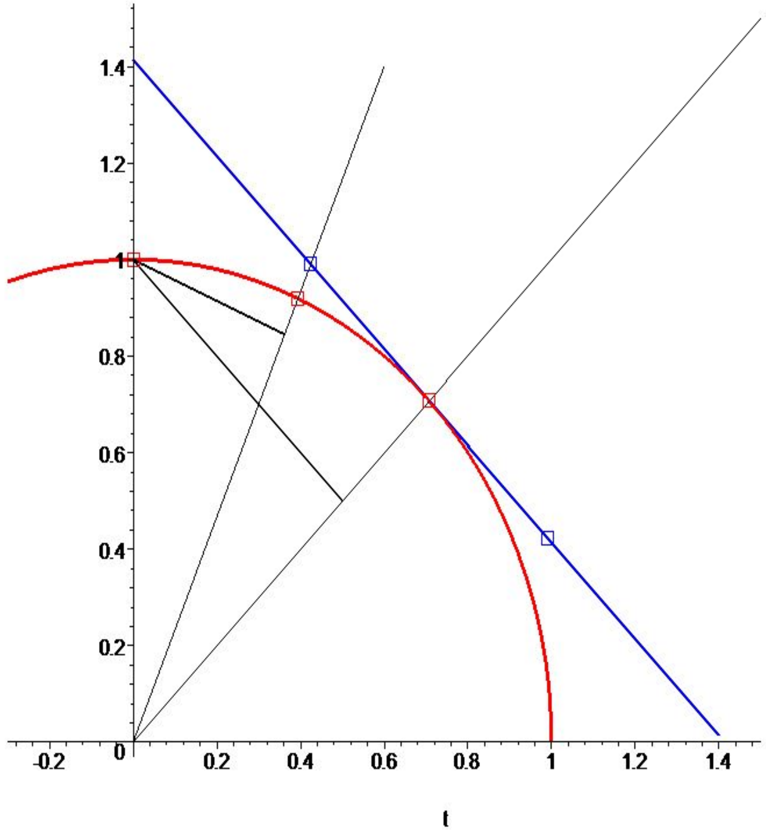
$$z - T_z f^{-1} f(z) \in T_z \mathbb{P}_n(\mathbb{C})$$

N.Op. Picture VIII



$$\pi(z - T_z f^{-1} f(z)) \in \mathbb{P}_n(\mathbb{C})$$

N.Op. Picture IX



$$\pi(z - T_z f^{-1} f(z)) \in \mathbb{P}_n(\mathbb{C})$$

* INPUT: *A System of Homogeneous Polynomials*

$$F := [f_1, \ldots, f_n] \in \mathcal{H}_{(d)},$$

$deg(f_i) = d_i$, $(d) := (d_1, \ldots, d_n)$.

A zero $\zeta \in V(F)$

An Approximate Zero(Smale'81) a point $z\mathbb{P}_n(\mathbb{C})$ such that Newton's operator N_F applied to z converges very fast to the zero.

$$d_T(N_F^k(z), \zeta) \leq \frac{1}{2^{2^{k-1}}}.$$

$d_T :=$ tangent "distance" .

Condition Number ([Shub–Smale, 86–96])

$$\mu_{norm}(F,\zeta) := \|F\| \|T_z F^{-1} \Delta(\|\zeta^{d_i - 1}\| d_i^{1/2})\|.$$

Condition Number Theorem : *Discriminant Variety in* $\mathbb{P}(\mathcal{H}_{(d)})$.

$$\Sigma_\zeta := \{F \in \mathbb{P}(\mathcal{H}_{(d)}) \; : \; \zeta \in V(F), T_\zeta F \notin GL(n, \mathbb{C})\}.$$

$$\Sigma := \bigcup_{\zeta \in \mathbb{P}_n(\mathbb{C})} \Sigma_\zeta. \quad \text{(Systems with a critical zero).}$$

Fiber Distance : $\rho(F,\zeta) := d_P(F, \Sigma_\zeta)$.

Theorem 5 (Shub-Smale, 91)

$$\mu_{norm}(F,\zeta) := \frac{1}{\rho(F,\zeta)}.$$

$$d := \max\{d_i \ : \ 1 \leq i \leq n\}.$$

Theorem 6 (Smale,81) *Si :*

$$d_T(z,\zeta) \leq \frac{3-\sqrt{7}}{d^{\frac{3}{2}}\mu_{norm}(F,\zeta)},$$

then, z is an approximate zero associated to some zero ζ of F.

NON–UNIVERSAL ALGORITHMICS

* INPUT:

A System $F \in \mathbb{P}(\mathcal{H}_{(d)})$,

* OUTPUT:

UNIVERSAL SOLVING: **An** *Approximate Zero* z **for each zero** $\zeta \in V(F)$.

Lower Complexity Bound: Bézout's Number ($\mathcal{D} := \prod_{i=1}^{n} d_i$) $\Rightarrow$ **Intractable**

Or:

NON-UNIVERSAL SOLVING : **An** *Approximate Zero* z **for some of the zeros** $\zeta \in V(F)$.

Complexity of Non–Universal Solving? (= Smale's 17th Problem)

Incidence Variety:

$$V := \{(F, \zeta) \in \mathbb{P}(\mathcal{H}_{(d)} \times \mathbb{P}_n(\mathbb{C}) \ : \ f(\zeta) = 0\}.$$

Two Canonical Projections:

$$\begin{array}{ccccc} & & V & & \\ \pi_1 & \swarrow & & \searrow & \pi_2 \\ \mathbb{P}(\mathcal{H}_{(d)}) & & & & \mathbb{P}_n(\mathbb{C}) \end{array}$$

Critical values of $\pi_1 = \Sigma$.

In fact, the following is a "covering map":

$$\pi_1 : V \setminus \Sigma' \longrightarrow \mathbb{P}(\mathcal{H}_{(d)}) \setminus \Sigma.$$

And the real codimension is: $codim_{\mathbb{P}(\mathcal{H}_{(d)})}(\Sigma) \geq 2.$

$\boxed{\text{NAMELY}}$

Except for a null measure subset, for each $F, G \in \mathbb{P}(\mathcal{H}_{(d)}) \setminus \Sigma$, :

$$[F, G] \cap \Sigma = \emptyset,$$

where

$$[F, G] := \{(1-t)F + tG, \quad t \in [0,1]\}.$$

and the following is also a "covering space":

$$\pi_1 : \pi_1^{-1}([F, G]) \longrightarrow [F, G].$$

Namely, for each $\zeta \in V(G)$ there is a curve:

$$\Gamma(F, G, \zeta) := \{(F_t, \zeta_t) \in V \ : \ \zeta_t \in V(F_t), t \in [0,1]\}.$$

An algorithmic Philosophy

Start at (G, ζ) ($t = 1$) and closely follow (by applying Newton's projective operator) a polygonal close to $\Gamma(F, G, \zeta)$ until you find an approximate zero of F.

Input $F \in \mathcal{H}_{(d)}$

With Initial Pair

$$(G, \zeta) \in \mathcal{H}_{(d)} \times \mathbb{P}_n(\mathbb{C}), \quad G(\zeta) = 0.$$

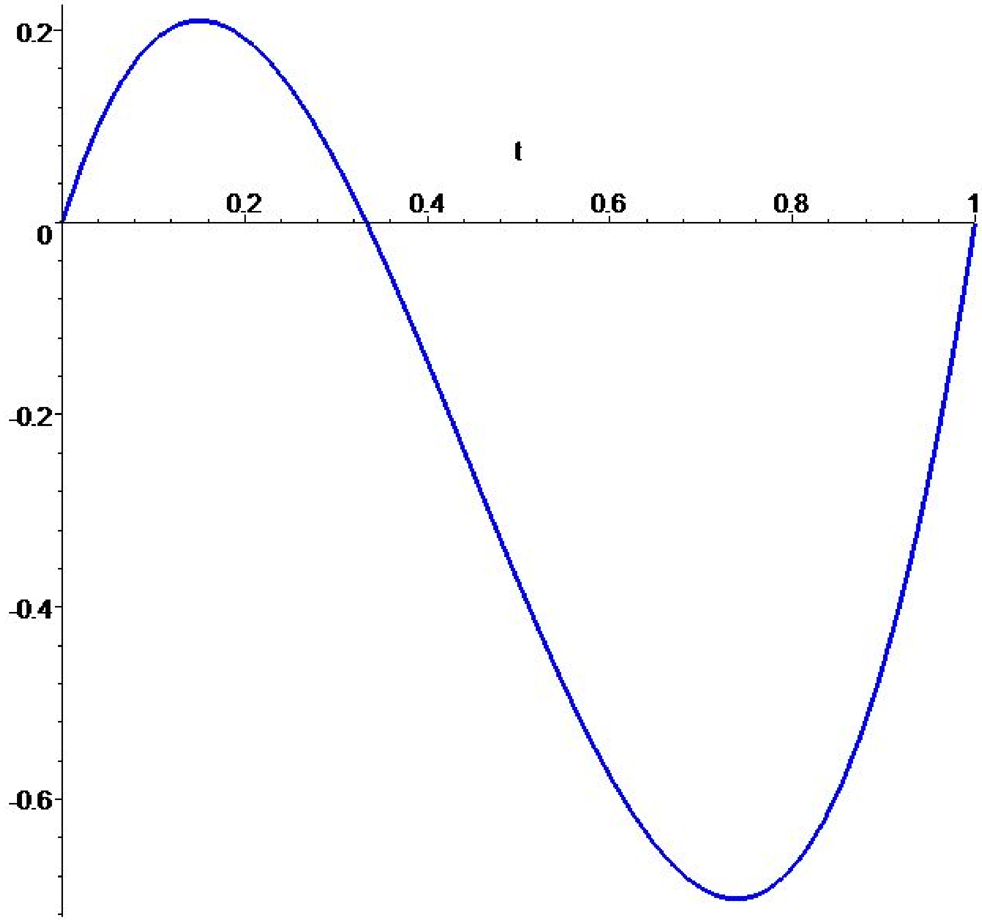
Following [F,G] and the cruve Γ

Output

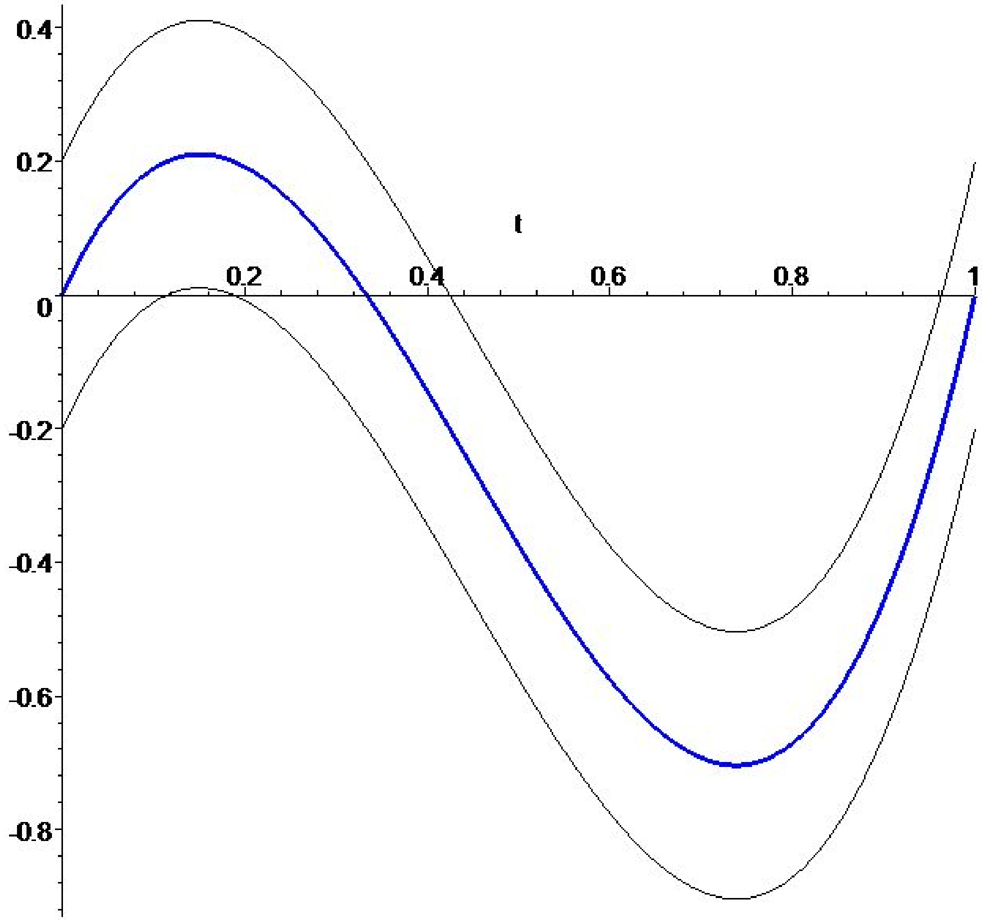
-- *Either ERROR*

-- *Or an approximate zero* $z \in \mathbb{P}_n(\mathbb{C})$ *associated to some zero* $\zeta \in \mathbb{P}_n(\mathbb{C})$ *of* $F \in \mathcal{H}_{(d)}$

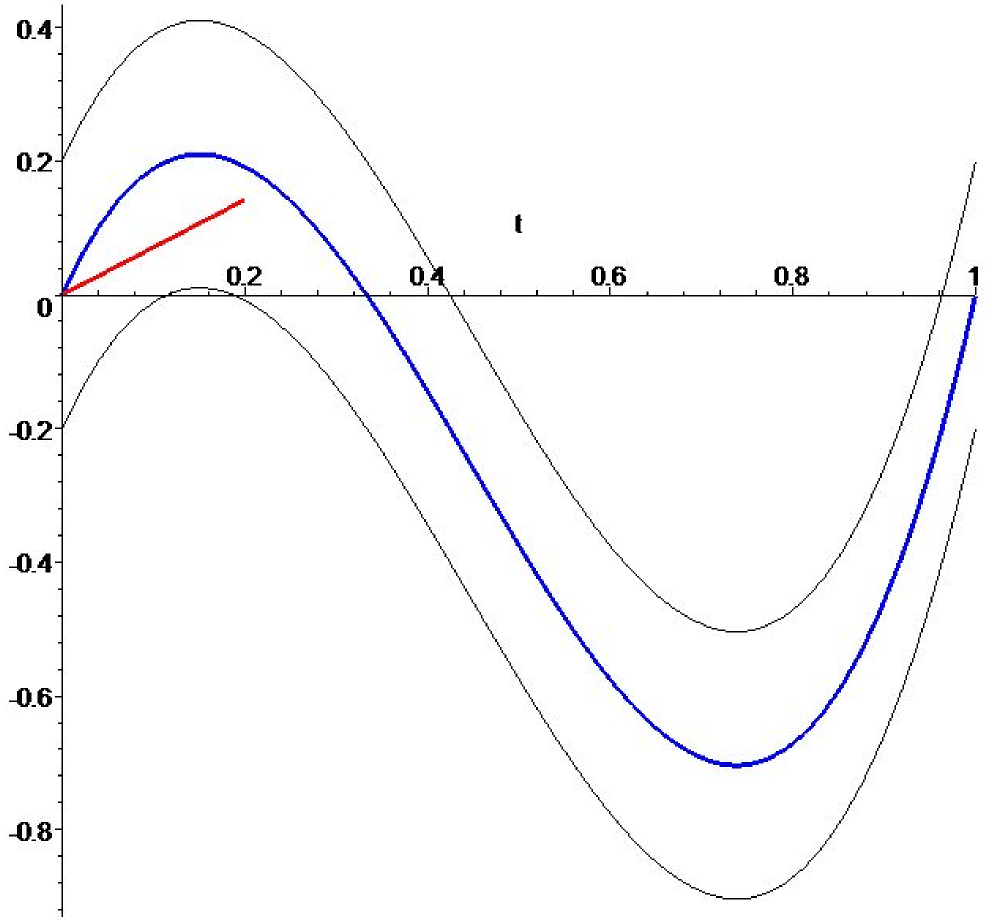
HD Picture I



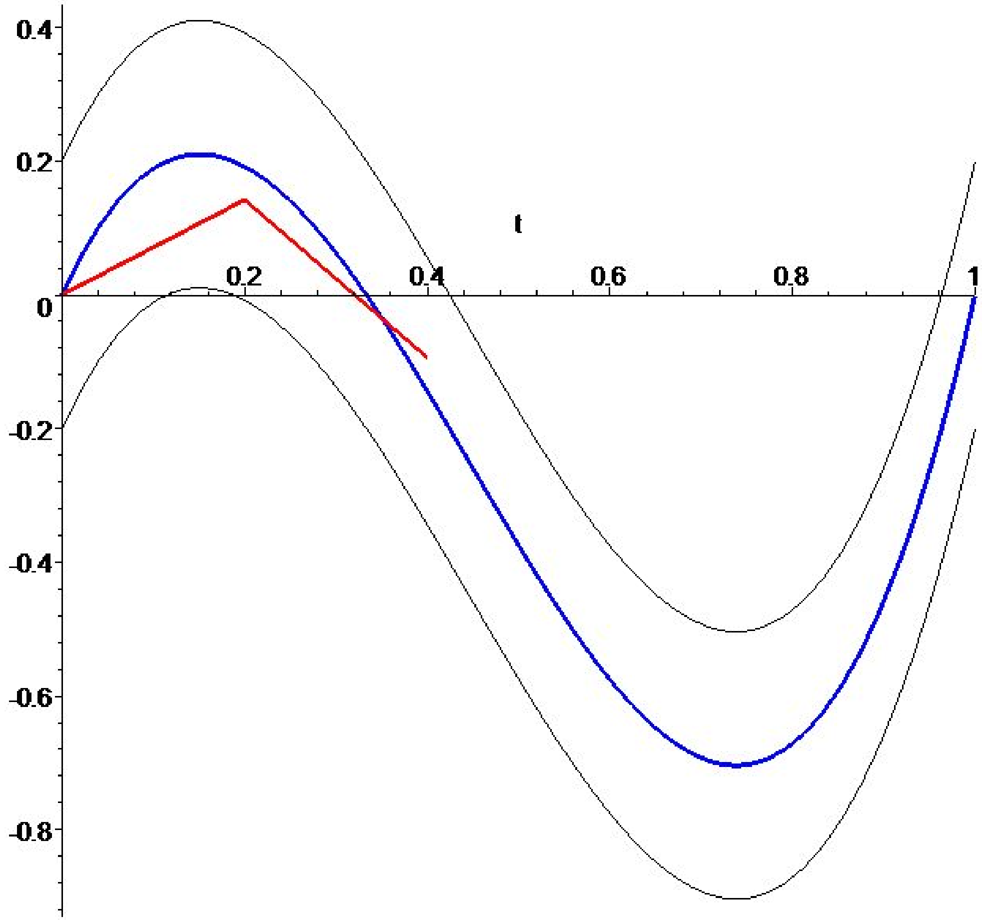
HD Picture II



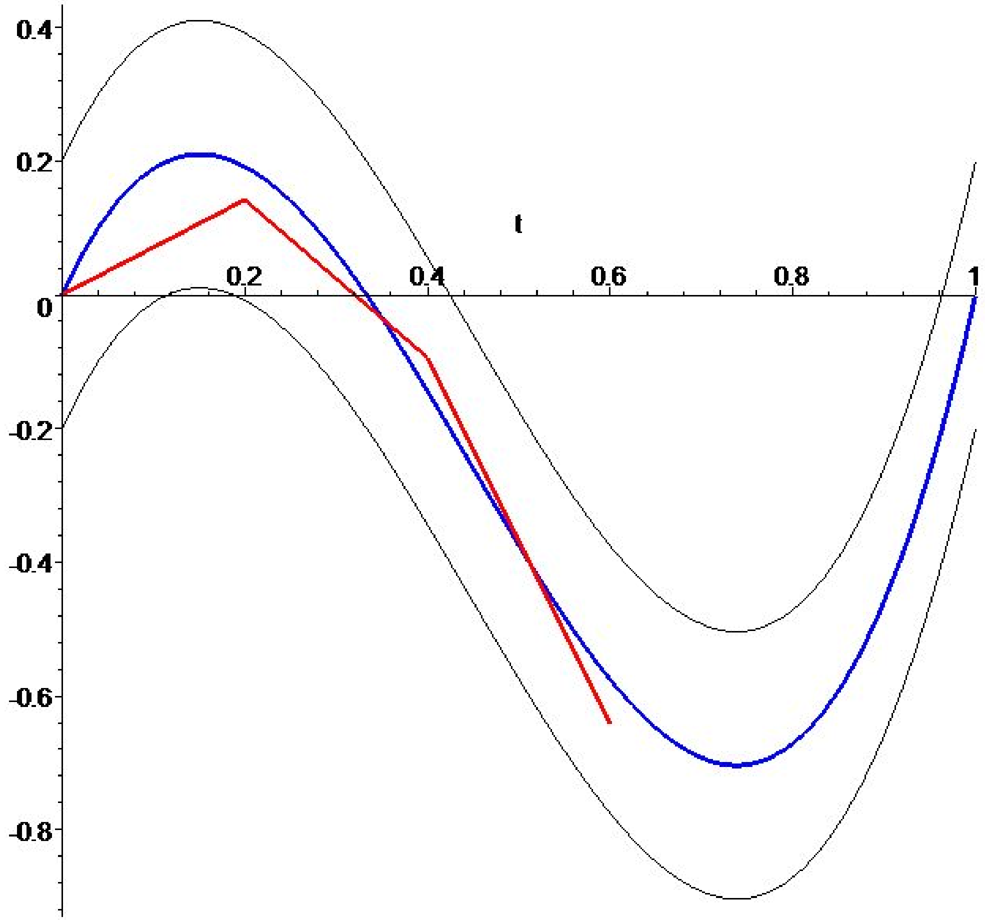
HD Picture III



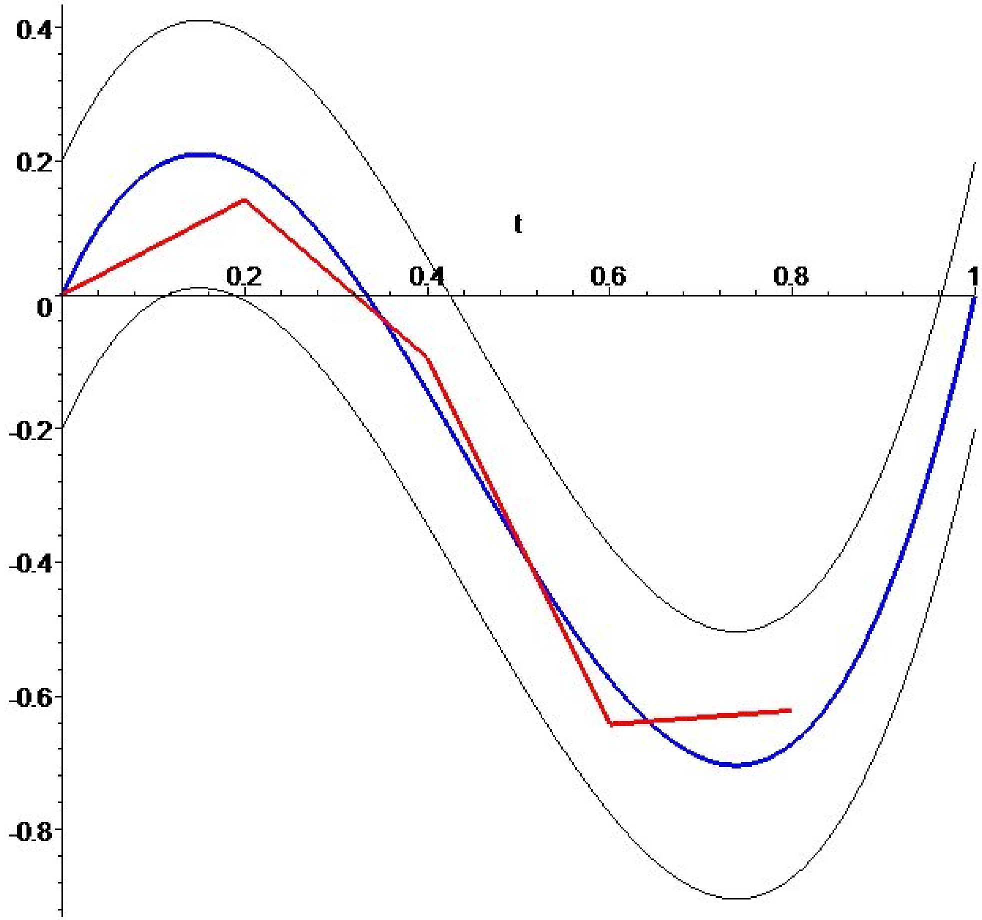
HD Picture IV



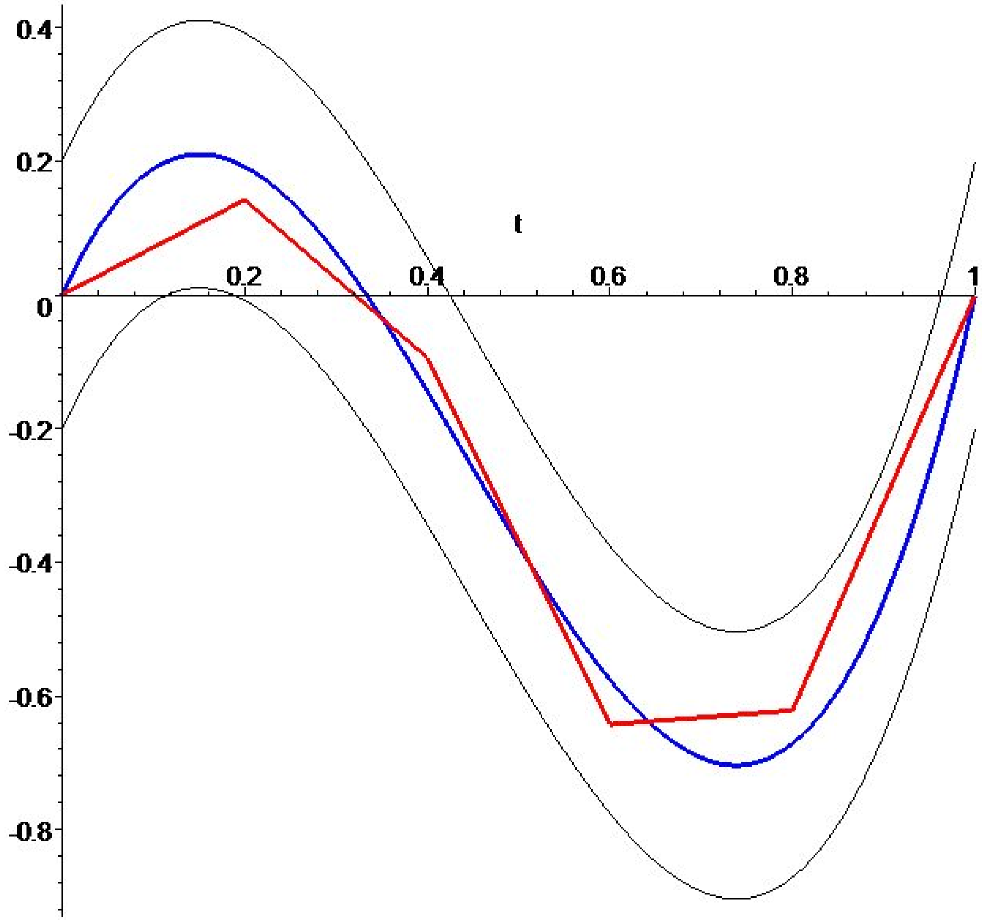
HD Picture V



HD Picture VI



HD Picture VII



THE PROBLEMS WITH THIS APPROACH (I)

Problem 1.- *What is the compexity of this method?*

Answer.–
– The complexity of each step is polynomial in the number of variables and the evaluation complexity of the input system. Thus, complexity mainly depends on the number of steps.

– The number of "homotopy steps" is bounded by $O(\mu_{norm}(\Gamma)^2)$ ([Shub-Smale, 91]), where

$$\mu_{norm}(\Gamma(F,G,\zeta)) := max\{\mu_{norm}(F_t,\zeta_t) \; : \; (F_t,\zeta_t) \in \Gamma(F,G,\zeta)\}.$$

The Problems with this approach (II)

Problem 2.- *worst case complexity is doubly exponential in the number of variables (voir exemple dans [castro–Hagele–Morais–P., 01]), and then?*

Answer.–

– "Worst case complexity" does not suffice to explain the behavior. Look at average complexity!.

– The word "average" forces to have some probability distribution, . *which one?.*

Answer (Sub-problem 2b).–
– The set $\mathbb{P}(\mathcal{H}_{(d)})$ is a complex and compact Riemannian manifold. Thus, it has an associated measure (a volume form in $d\nu_{\mathbb{P}}$) such that the volume $\nu_{\mathbb{P}}[\mathbb{P}(\mathcal{H}_{(d)})]$ is finite. Then we also have a probability distribution.

– The probability measure in $\mathbb{P}(\mathcal{H}_{(d)})$ equivalent to Gaussian distribution in the affine space $\mathcal{H}_{(d)}$.

Sub–problem 2c.–*Since computing is discrete, what is the distribution for discrete inputs (namely polynomials with coefficients in a discrete field)?.*

The Problems with this approach (IV)

Problem 3.– *Anyway, this approach is not defining an algorithm (since we have o initial pair). Is there a true algorithm of polynomial average complexity?.*

Answers.–

1.– Yes.

2.– Polynomial in the dimension of the space of inputs (dense encoding of polynomials).

Once again HD

Input $F \in \mathcal{H}_{(d)}$

Apply homotopic defomration (HD) *with initial pair*

$$(G, z) \in \mathcal{H}_{(d)} \times \mathbb{P}_n(\mathbb{C})$$

following the curve $\Gamma(F, G, z)$ *of* $\Gamma = \pi_1^{-1}([F, G]))$ *that contains* (G, z).

Output:

– *Either ERROR*

– *or an approximate zero of* F.

HD with resources bounded by a function $\varphi(f, \varepsilon)$.

INPUT $F \in \mathcal{H}_{(d)}$, $\varepsilon > 0$

Perform $\varphi(f, \varepsilon)$ steps of homotopic deformation (HD) *with initial pair*

$$(G, z) \in \mathcal{H}_{(d)} \times \mathbb{P}_n(\mathbb{C})$$

following the curve $\Gamma(F, G, z)$ *in* $\Gamma = \pi_1^{-1}([F, g]))$ *that contains* (G, z).

OUTPUT:

– *Either ERROR*

– *or an approximate zero of* F.

$\varepsilon-$EFFICIENT INITIAL PAIRS

Definition *A pair* $(G, \zeta) \in V$ *is* $\varepsilon-$*efficient if the resources function for the resources:*

$$\varphi(f, \varepsilon) := 10^5 n^5 N^2 d^3 \varepsilon^{-2}.$$

For randomly chosen input system $F \in \mathbb{P}(\mathcal{H}_{(d)})$ *the algorithm HD with initial pair* (G, z) *and resources bound* φ *outputs un approximate zero of* F *with probability greater than:*

$$1 - \varepsilon.$$

HD WITH $\varepsilon-$EFFICIENT INITIAL PAIR

Let $(G_\varepsilon, \zeta_\varepsilon)$ be an $\varepsilon-$efficient initial pair.

INPUT $F \in \mathcal{H}_{(d)}$, $\varepsilon > 0$

Perform $\varphi(f, \varepsilon)$ steps of HD *with initial pair*

$$(G_\varepsilon, \zeta_\varepsilon) \in \mathcal{H}_{(d)} \times \mathbb{P}_n(\mathbb{C})$$

following $\Gamma(F, G_\varepsilon, \zeta_\varepsilon)$.

OUTPUT:

– *Either ERROR*

– *or an approximate zero of* F.

Theorem 7 *([Shub-Smale, BezV, Beltrán-P, Bez V 1/2] There exist $\varepsilon-$efficient initial pairs.*

Remark 8 *Even with $\zeta_\varepsilon = (1 : 0 : \cdots : 0)$.*

Smale 17th Problem.– How to construct $\varepsilon-$efficient initial pairs?.

Questor Sets

[Beltrán- P., 2006]

A subset $\mathcal{G} \subseteq V$ (incidence variety) is *a questor set for HD if:*

for every $\varepsilon > 0$ *the probability that a randomly chosen pair* $(G, \zeta) \in \mathcal{G}$ *is* $\varepsilon-$*efficient for HD is greater than*

$$1 - \varepsilon.$$

INPUT $F \in \mathcal{H}_{(d)}, \varepsilon > 0$

Guess at random $(G, \zeta) \in \mathcal{G}$

Apply $\varphi(f, \varepsilon)$ *deformation steps HD between* G *and* F*, starting at* (G, ζ)*.*

OUTPUT:

– *Either ERROR (with probability smaller than* ε*)*

– *or un approximate zero of* F *(with probability greater than* $1 - \varepsilon$*).*

COMMENTS

Minor: **It is a probabilistic algorithm**

Relevant: **The questor set $\mathcal{G}$ must be easy to construct and easy t handle .**

Theorem[*Beltrán, P. 2006*] *We succeeded to exhibit a constructible and easy to mhandle questor set.*

$e := (1 : 0 : \ldots : 0) \in \mathbb{P}_n(\mathbb{C})$ *a "pole" in the complex sphere.*

$V_e := \{F \in \mathcal{H}_{(d)} \ : \ F(e) = 0\}$. *Systems vanising at the "pole" e.*

$F \in V_e \longmapsto F : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^n$.

The tangent mapping $T_eF := DF(e)$ *restricted to the tangent space* $T_e\mathbb{P}_n(\mathbb{C}) = e^{\perp} = \mathbb{C}^n \subseteq \mathbb{C}^{n+1}$.:

$$T_eF := T_e\mathbb{P}_n(\mathbb{C}) = \mathbb{C}^n \longrightarrow \mathbb{C}^n.$$

A First Approach

$L_e := \{F \in V_e \, : \, T_e F = F\}$. *"linear part" of the systems in* V_e.

$L_e^{\perp}$:= *Systems in* V_e *of order greater than 2 at* e.

Remark.- *$V_e, L_e, L_e^{\perp}$ are linear subspaces of $\mathcal{H}_{(d)}$ given by their coefficient list.*

Naïve Idea: *Consider*

$$\mathcal{G} := \{(G, e) \, : \, G \in V_e = L_e^{\perp} \bigoplus L_e\}.(?)$$

Towards Questor Sets II (L_e)

$\mathcal{U}(n+1)$:= *unitary matrices defined in* $\mathbb{C}^{n+1}$.

$\mathcal{H}_{(1)} := \mathcal{M}_{n\times n+1}(\mathbb{C})$ **space of $n \times (n+1)$ complex matrices.**

$$X^{(d)} := \begin{pmatrix} X_0^{d_1-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_0^{d_n-1} \end{pmatrix}.$$

$$V_e^{(1)} := \{(M,U)\ :\ M \in \mathcal{H}_{(1)}, U \in \mathcal{U}, UKer(M) = e\}.$$

$\boxed{\text{LINER PART } L_e}$

$$\psi_e : V_e^{(1)} \qquad \longrightarrow \qquad L_e$$

$$\psi_e(M, U) := X^{(d)}(MU) \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}.$$

$$L_e := Im(\psi_e(M, U)).$$

A useful constant

$$T := \left(\frac{n^2+n}{N}\right)^{n^2+n} \in \mathbb{R}, \qquad t \in [0, T].$$

Towards a questor set III

$$\mathbb{G} := [0, T] \times L_e^{\perp} \times V_e^{(1)}.$$

$$G : \mathbb{G} \longrightarrow V_e,$$

$$(t, L, M, U) \in \mathbb{G} \longmapsto G(t, L, M, U) \in V_e$$

$$G(t, L, M, U) := (1 - t^{\frac{1}{n^2+n}})^{1/2} L + t^{\frac{1}{n^2+n}} \psi_e(M, U) \in V_e,$$

Theorem 9 (Beltrán-P., 2005a) *For every degree list* $(d) := (d_1, \ldots, d_n)$, *the set*

$$\mathcal{G}_{(d)} := Image(G) = G(\mathbb{G}).$$

is questor set of initial pairs for HD. Namely,

A system $(G, e) \in \mathcal{G}_{(d)}$ chosen at random is $\varepsilon-$efficient for HD with probability greater than

$$1 - \varepsilon.$$

THE ALGORITHM

INPUT: $F \in \mathcal{H}_{(d)}$, $\varepsilon > 0$.

Guess at random $(G, e) \in \mathcal{G}_{(d)}$ $(Guess\ (t, L, M)\ldots)$

Apply $\varphi(F, \varepsilon)$ **homotopic deformation steps**

OUTPUT: Either "ERROR" or an approximate zero z of F.

Meaning (I)

Theorem 10 [Beltrán-P,06] *There is a probabilistic algorithm (bounded error probability) for non–universal projective solving of systems of homogeneous polynomial equations such that for every positive real number* $\varepsilon > 0$*:*

- *The running time of the algorithm is at most:*

$$O(n^5 N^2 \varepsilon^{-2})$$

- *The probability that the algorithm outputs an approximate zero is greater than:*

$$1 - \varepsilon$$

Corollary 11 [Beltrán-P,06] *There is a probabilistic algorithm (bounded error probability) for non–universal projective solving of systems of homogeneous polynomial equations of degree 3 such that for every positive real number $\varepsilon > 0$:*

- *The running time of the algorithm is at most:*

$$O(n^{13}\varepsilon^{-2})$$

- *The probability that the algorithm outputs an approximate zero is greater than:*

$$1 - \varepsilon$$

Remarque Taking $\varepsilon = 1/n^2$, the algorithm computes approximate zeros with probability greater than

$$1 - 1/n^2.$$

in time

$$O(n^{15}).$$

In [Beltrán-P., 07] we slighty modified our algorithm to get average complexity:

Definition 12 (Strong Questor Set) *A subset $\mathcal{G} \subseteq V$ is a strong questor set if*

$$E_{\mathcal{G}}[A_\varepsilon] \leq 10^4 n^5 N^3 d^{3/2} \varepsilon^2,$$

where

$$A_\varepsilon(G, z) := Prob_{\mathbb{P}(\mathcal{H}_{(d)})}[\mu_{norm}(F, G, z) > \varepsilon^{-1}].$$

Strong Questor Set

Theorem 13 (Beltrán-P.,07) *For every strong questor set $\mathcal{G}$, there is a measurable subset $\mathcal{C}$ such that the following holds:*

$$Prob_{\mathcal{G}}[\mathcal{C}] \geq 4/5.$$

For every $\varepsilon > 0$ and for every $(G, z) \in \mathcal{C}$, (G, z) is a $\varepsilon-$efficient initial pair.

Theorem 14 (Beltrán-P.,07) *The set $\mathcal{G}_{(d)}$ is a strong questor set.*

Corollary 15 *There is a bounded error probability algorithm of average polynomial time that for all but a zero measure subset of systems of homogeneous polynomial equations computes projective approximate zeros .*

By average complexity we mean:

$$E_{\mathbb{P}(\mathcal{H}_{(d)})}[T_{\mathcal{P}}] := \frac{1}{\nu_{\mathbb{P}}[\mathbb{P}(\mathcal{H}_{(d)})]} \int_{\mathbb{P}(\mathcal{H}_{(d)})} T_{\mathcal{P}}(f) d\nu_{\mathbb{P}} = O(n^5 N^3),$$

$T_{\mathcal{P}}(f) :=$ *running time on input* f.

[Beltrán-P., 07] :

Corollary 16 *There is a bounded error probability algorithm of average polynomial time that for all but a zero measure subset of systems of homogeneous polynomial equations computes affine approximate zeros.*

By average complexity we mean:

$$E_{\mathbb{P}(\mathcal{H}_{(d)})}[T_{\mathcal{A}}] := \frac{1}{\nu_{\mathbb{P}}[\mathbb{P}(\mathcal{H}_{(d)})]}\int_{\mathbb{P}(\mathcal{H}_{(d)})} T_{\mathcal{A}}(f)d\nu_{\mathbb{P}} = O(N^5),$$

$T_{\mathcal{A}}(f) :=$*running time on input* f.

[Beltrán-P., 07]

Theorem 17 *Let $\delta > 0$ be a positive real number. For every $F \in \mathbb{P}(\mathcal{H}_{(d)})$, let*

$$V_A(F) := \{x \in \mathbb{C}^n \ : \ f(x) = 0\},$$

be the set of affine solutions Let

$$||V_A(F)|| := sup\{||x|| \ : \ x \in V_A(F)\} \in [0, \infty],$$

the maximal norm of its zeros.

Then, the probability that for a randomly chosen affine system $F \in \mathbb{P}(\mathcal{H}_{(d)})$ we have $||V_A(F)|| > \delta$ is at most:

$$\mathcal{D}\sqrt{\pi n}\delta^{-1}$$

.

In fact, we proved:

$$E_{\mathbb{P}(\mathcal{H}_{(d)})}[||V_A(f)||] = \mathcal{D}\frac{\Gamma(1/2)\Gamma(n+1/2)}{\Gamma(n)} \leq \mathcal{D}\sqrt{\pi n}.$$

Immediate Open Questions

Real Solving ?: Zero–dimensional Case.

Singular Zeros: Homotopy Techniques?.

Adaptability to Other Input Data Structures: Does it work for straight–line programa input structure?.