The probability that a slight perturbation of a numerical analysis problem is difficult

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Outline

Introduction

Condition Number: Definition

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More specifically, after fixing norms $\|\cdot\|$ on $\mathbb{R}^p$ and $\mathbb{R}^m$, we define

$$\kappa(f, a) := \limsup_{\Delta a \to 0} \frac{\|f(a+\Delta a) - f(a)\| / \|f(a)\|}{\|\Delta a\| / \|a\|}.$$

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If f is differentiable, we have

$$\kappa(f,a) = \|Df(a)\| \frac{\|a\|}{\|f(a)\|},$$

where $\|Df(a)\|$ denotes the operator norm of the Jacobian of f at a.

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hence

$$\frac{\|\Delta x\|}{\|x\|} \frac{\|A\|}{\|\Delta A\|} \leq \kappa(A) + \mathcal{O}(\|\Delta A\|).$$

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  - Shub and Smale’s Newton homotopy method to solve systems of polynomial equations

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- Hence worst-case analysis does not make sense in this context.
- An average-case analysis of the running time of a numerical algorithm reduces to an analysis of the distribution (or expected value) of the condition number for random inputs a.
  Cf. Demmel, Edelman, Kostlan, Renegar, Shub, Smale,...

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Φ distribution on $\mathbb{R}^p$, $N(0, \sigma^2)$ Gaussian distribution.

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Geometric Interpretation of Condition Numbers

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The Eckart-Young Theorem states that

$$\|A^{-1}\| = \frac{1}{\mathrm{dist}(A,\Sigma)},$$

where dist refers to the Frobenius norm $\|A\|^2 := \sum_{ij} a_{ij}^2$ (coming from the canonical scalar product on $\mathbb{R}^{n\times n}$).

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The modified condition number

$$\kappa_F(A) := \|A\|_F \, \|A^{-1}\| = \frac{\|A\|_F}{\text{dist}(A, \Sigma)},$$

differs from $\kappa(A)$ at most by a factor of $\sqrt{n}$.

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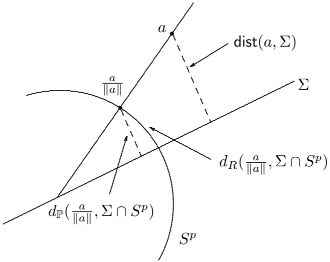
  where the norm and distance are induced by the canonical inner product.
- Let $\Sigma \subseteq \mathbb{R}^{n \times n}$ be the set of singular matrices A. The condition number $\kappa_F(A)$ is conic by the Eckart-Young Theorem:

$$\kappa_F(A) = \|A\|_F \|A^{-1}\| = \frac{\|A\|_F}{\text{dist}_F(A, \Sigma)}.$$

Conic Condition Numbers 2

$$\mathscr{C}(a) = \frac{\|a\|}{\mathsf{dist}(a,\Sigma)} = \frac{1}{d_{\mathbb{P}}(\frac{a}{\|a\|}, \Sigma \cap S^p)},$$

where $d_{\mathbb{P}}(x,y) = \sin d_R(x,y)$ denotes the projective distance on the sphere $S^p := \{x \in \mathbb{R}^{p+1} \mid \|x\| = 1\}$. Hence, we may restrict to data a lying on the sphere S^p.



Uniform Smoothed Analysis

- For $a \in S^p$ and $\sigma > 0$ let

$$B(a, \sigma) := \{z \in S^p \mid d_{\mathbb{P}^p}(z, a) \leq \sigma\}.$$

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- Uniform smoothed analysis of a conic condition number $\mathscr{C}$ is the study of
$$\sup_{a \in S^p} \mathop{\mathbf{E}}_{z \in B(a,\sigma)} \ln \mathscr{C}(z),$$
where $z \in B(a,\sigma)$ means that z is uniformly distributed in $B(a,\sigma)$.

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  - $\sigma = 0$ yields worst-case analysis
  - $\sigma = 1$ yields average-case analysis
- For any $a \in S^p$ and $\sigma > 0$, we are thus interested in studying the distribution of the random variables $\mathscr{C}(z)$ and $\ln \mathscr{C}(z)$ on $B(a,\sigma)$.

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Theorem

Let $\mathscr{C}$ be a conic condition number with set of ill-posed inputs Σ. Assume that $\Sigma \cap S^p \subseteq W$ where $W \subset S^p$ is the zero set in S^p of homogeneous polynomials of degree at most d.
Then, for all $\sigma \in (0,1]$ and all $t \geq (2d+1)\frac{p}{\sigma}$,

$$\sup_{a \in S^p} \underset{z \in B(a,\sigma)}{\mathrm{Prob}} \{\mathscr{C}(z) \geq t\} \leq 26\, dp\, \frac{1}{\sigma t}.$$

and

$$\sup_{a \in S^p} \underset{z \in B(a,\sigma)}{\mathbf{E}} (\ln \mathscr{C}(z)) \leq 2 \ln p + 2 \ln d + 2 \ln \frac{1}{\sigma} + 4.7.$$

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Similar bounds on the tail of the distribution of $\mathscr{C}(z)$ in the framework of average case analysis ($\sigma = 1$) in terms of these parameters have been given by Demmel (1988) and by Beltrán-Pardo (2005) (over $\mathbb{C}$).

Applications

Linear Equation Solving

- Problem: Solving the system of equations $Ax = b$, $A \in \mathbb{R}^{n \times n}$.
- Set of ill-posed inputs: $\Sigma = \{A \in \mathbb{R}^{n \times n} \mid \det A = 0\}$
- Condition number: $\kappa_F(A) = \|A\|_F \|A^{-1}\|$.

Corollary

For all $A \in \mathbb{R}^{n \times n}$ of Frobenius norm one and $0 < \sigma \leq 1$ we have

$$\mathop{\mathbf{E}}_{Z \in B(A,\sigma)} (\ln \kappa_F(Z)) \leq 6 \ln n + 2 \ln \frac{1}{\sigma} + 4.7.$$

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M. Wschebor derived similar bounds for Gaussian perturbations by direct methods (2004).

Eigenvalue Computations

- Problem: Compute the (complex) eigenvalues of a matrix $A \in \mathbb{R}^{n \times n}$.
- Set of ill-posed inputs: Matrices A having multiple eigenvalues.
- Condition number (Wilkinson): Satisfies $\kappa_{\text{eigen}}(A) \leq \frac{\sqrt{2}\,\|A\|_F}{\text{dist}(A,\Sigma)}$.

Corollary

For all $A \in \mathbb{R}^{n \times n}$ and $0 < \sigma \leq 1$ we have

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Proof: $W = \Sigma$ is the zeroset of the discriminant of the characteristic polynomial, which has degree $d = n^2 - n$. Apply the theorem. □

Complex Polynomial Systems

- Fix $d_1, \dots, d_n \in \mathbb{N} \setminus \{0\}$. We denote by $\mathcal{H}_{\mathbf{d}}$ the vector space of polynomial systems $f = (f_1, \dots, f_n)$ with $f_i \in \mathbb{C}[X_0, \dots, X_n]$ homogeneous of degree d_i. $\mathcal{H}_{\mathbf{d}}$ carries an invariant Hermitian product.

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- In a seminal series of papers, M. Shub and S. Smale studied the problem of, given $f \in \mathcal{H}_{\mathbf{d}}$, compute an approximation of a complex zero of f. They proposed an algorithm and studied its complexity in terms of a condition number $\mu_{\mathrm{norm}}(f)$ for f.

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Corollary

For all $f \in \mathcal{H}_{\mathbf{d}}$ of norm one we have

$$\mathop{\mathbf{E}}_{g \in B(f,\sigma)} (\ln \mu_{\mathrm{norm}}(g)) \leq 2 \ln N + 4 \ln \mathcal{D} + 2 \ln n + 2 \ln \frac{1}{\sigma} + 6.1.$$

where $N = \dim \mathcal{H}_{\mathbf{d}} - 1$ and $\mathcal{D} = d_1 \cdots d_n$ is the Bézout number.

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S. Smale and M. Shub obtained similar estimates for average complexity.

Outline of Proof of Main Theorem

Geometric Reformulation of the Problem

Given a conic condition number $\mathscr{C}$ with ill-posed set Σ. Let $W \subset S^p$ such that $\Sigma \cap S^p \subseteq W$. Then for $z \in S^p$

$$\mathscr{C}(z) \geq \frac{1}{\varepsilon} \Longleftrightarrow d_{\mathbb{P}^p}(z, \Sigma \cap S^p) \leq \varepsilon \Rightarrow d_{\mathbb{P}^p}(z, W) \leq \varepsilon.$$

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We denote by $T(W, \varepsilon)$ the ε-neighborhood around W in S^p:

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Then:

$$\operatorname*{Prob}_{z \in B(a,\sigma)} \left\{ \mathscr{C}(z) \geq \frac{1}{\varepsilon} \right\} \leq \operatorname*{Prob}_{z \in B(a,\sigma)} \{ d_{\mathbb{P}^p}(z, W) \leq \varepsilon \} = \frac{\mathrm{vol}(T(W, \varepsilon) \cap B(a, \sigma))}{\mathrm{vol}(B(a, \sigma))}.$$

Geometric Version of Main Result

Let $\mathcal{O}_p := \mathrm{vol}(S^p) = \frac{2\pi^{\frac{p+1}{2}}}{\Gamma\left(\frac{p+1}{2}\right)}$ denote the volume of S^p.

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The main theorem follows from the following purely geometric statement on the volume of patches of tubes around subvarieties in spheres.

Theorem'

Let $W \subset S^p$ be a real algebraic variety defined by homogeneous polynomials of degree at most $d \geq 1$. Then we have for $a \in S^p$ and $0 < \varepsilon, \sigma \leq 1$

$$\frac{\mathrm{vol}\left(T(W,\varepsilon) \cap B(a,\sigma)\right)}{\mathrm{vol} B(a,\sigma)} \leq 4 \sum_{k=1}^{p-1} \binom{p}{k} (2d)^k \left(1 + \frac{\varepsilon}{\sigma}\right)^{p-k} \left(\frac{\varepsilon}{\sigma}\right)^k + \frac{p\mathcal{O}_p}{\mathcal{O}_{p-1}} (2d)^p \left(\frac{\varepsilon}{\sigma}\right)^p.$$

The three main steps of the proof

- Upper bound on the volume of an ε-neighborhood of a smooth hypersurface in terms of integrals of absolute curvature.
  This is a variation of H. Weyl's exact formula for the volume of tubes, a formula which, however, only holds for sufficiently small ε.

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This is based on the kinematic formula of integral geometry and Bézout's theorem.

III Remove the smoothness assumption by some perturbation argument.

Some differential geometry of hypersurfaces on spheres

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- For $1 \leq i < p$ we define the ith curvature $K_{M,i}(x)$ of M at x as the ith elementary symmetric polynomial in $\kappa_1(x), \ldots, \kappa_{p-1}(x)$, and put $K_{M,0}(x) := 1$.

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- Note $|\mu_0|(U) = \mathrm{vol}(U)$.

Bounding the volume of tubes

The ε-tube $T^{\perp}(M, \varepsilon)$ around M is a subset of $T(M, \varepsilon)$ obtained by "cutting off points close to the boundary" of M:

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Variant of Weyl's Tube Formula

Let M be a compact, oriented, smooth hypersurface of S^p and U be an open subset of M. For all $0 < \varepsilon \leq 1$

$$\mathrm{vol}(T^{\perp}(U,\varepsilon)) \leq 2\sum_{i=0}^{p-1} J_{p,i+1}(\varepsilon)\cdot|\mu_i|(U),$$

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The leading term is $2J_{p,1}(\varepsilon)\cdot|\mu_0|(U) \approx 2\varepsilon\,\mathrm{vol}(U)$.

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Let $f \in \mathbb{R}[X_0, \ldots, X_p]$ be homogeneous of degree $d > 0$ with nonempty zero set $V \subseteq S^p$ such that the derivative of the restriction of f to S^p does not vanish on V. Then V is a compact smooth hypersurface of S^p.

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For $a \in S^p$, $0 < \sigma \leq 1$, and $0 \leq i < p$ we have

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Proof ingredients:

- Principal kinematic formula of integral geometry for spheres
- Bézout's theorem

Crofton's formula from integral geometry

We denote by dG the volume element on the orthogonal group $G = O(p+1)$ (compact Lie group), normalized such that the volume of G equals one. G operates on S^p in the natural way.

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Let T be a submanifold of S^p with dim $T = p - 1$. Then

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- This allows to bound $\mathrm{vol}(V \cap B_{\mathbb{P}}(a,\sigma))$ via Bézout's theorem.
- To bound μ_i for $i > 0$ we need a more sophisticated tool.

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We will need the following version of this formula for spheres (cf. Santaló, Howard).

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Let U be an open subset of a compact oriented smooth hypersurface M of S^p and $0 \leq i < p-1$. Then we have

$$\mu_i(U) = \mathcal{C}(p,i) \int_{g \in G} \mu_i(gU \cap S^{i+1})\, dG(g),$$

where $\mathcal{C}(p,i) = (p-i-1)\binom{p-1}{i} \frac{\mathcal{O}_{p-1}\mathcal{O}_p}{\mathcal{O}_i \mathcal{O}_{i+1} \mathcal{O}_{p-i-2}}$.

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This allows to reduce the estimation of μ_i to the case of codimension one, i.e., Gaussian curvature, which can be treated directly.

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