Infinite Global Fields and the Generalized Brauer–Siegel Theorem

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To our teacher Yu.I.Manin on the occasion of his 65th birthday

Abstract. The paper has two purposes. First, we start to develop a theory of infinite global fields, i.e., of infinite algebraic extensions either of $\mathbb{Q}$ or of $\mathbb{F}_r(t)$. We produce a series of invariants of such fields, and we introduce and study a kind of zeta-function for them. Second, for sequences of number fields with growing discriminant we prove generalizations of the Odlyzko–Serre bounds and of the Brauer–Siegel theorem, taking into account non-archimedean places. This leads to asymptotic bounds on the ratio $\log hR/\log\sqrt{|D|}$ valid without the standard assumption $n/\log\sqrt{|D|}\to 0$, thus including, in particular, the case of unramified towers. Then we produce examples of class field towers, showing that this assumption is indeed necessary for the Brauer–Siegel theorem to hold. As an easy consequence we ameliorate on existing bounds for regulators.

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1 Introduction

A global field K is a finite algebraic extension either of the field $\mathbb{Q}$ of rational numbers, or of the field $\mathbf{Q}_r=\mathbb{F}_r(t)$ of rational functions in one variable over a finite field of constants. An *infinite global field* $\mathcal{K}$ is either an infinite algebraic extension of $\mathbb{Q}$, or such an infinite algebraic extension of $\mathbf{Q}_r$ that $\mathcal{K}\cap\bar{\mathbb{F}}_r=\mathbb{F}_r$. In the first case we call $\mathcal{K}$ an *infinite number field*, in the second an *infinite function field over* $\mathbb{F}_r$.

The first *raison d'être* of our paper is an attempt to convince ourselves and the reader that there exists a (not yet constructed) non-trivial theory of such fields. In particular, we produce a series of invariants, and introduce and study a kind of zeta-function of such a field.

The second one is much more down to earth. For sequences of number fields

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with growing discriminant we prove generalizations of the Odlyzko–Serre bounds and of the Brauer–Siegel theorem, taking into account non-archimedean places. This leads to asymptotic bounds on the ratio $\log hR/\log\sqrt{|D|}$ valid without the standard assumption $n/\log\sqrt{|D|}\to 0$, thus including, in particular, the case of unramified towers. Then we produce examples of class field towers, showing that this assumption is indeed necessary for the Brauer–Siegel theorem to hold.

Wanting to study infinite global fields, we should think about examples. For "large" fields like $\bar{\mathbb{Q}}$ or $\mathbb{Q}^{\mathrm{ab}}$ the invariants we find are trivial, but there are numerous "smaller" ones, like the limit (i.e., the union) of fields of a given unramified (or "not too much" ramified) tower of fields. It is for these smaller ones that the theory we start to develop below is interesting.

To start with, an infinite global field is always the limit of a tower of finite ones:

$$\mathcal{K} = \operatorname*{ind\,lim}_{i\to\infty} K_i = \bigcup_{i=1}^{\infty} K_i, \text{ where } K_1 \subset K_2 \subset K_3 \subset \dots .$$

This tower is, of course, not unique. We are looking for invariants of $\mathcal{K}$, i.e., for parameters of $K_1 \subset K_2 \subset K_3 \subset \dots$ that do not depend on the tower, but only on its limit.

We use the following notation: Let $\{K_i\}$ for $i = 1, 2, \dots$ be a sequence of pairwise non-isomorphic global fields, either number or function; we set

$$g_i = \mathrm{genus}(K_i)$$

in the function field case, and

$$g_i = \log\sqrt{|D_i|}$$

in the number field case; we call it the *genus* of a number field.

Attention: Here and below we use the following agreement. In the number field case notation log means the natural logarithm $\log_e$. In the function field case over $\mathbb{F}_r$ the same notation log means $\log_r$. As we shall see below, this is justified by the uniformity of results obtained.

One of the reasons to think that the definition of genus for the number field case is natural is that for any given g_0 there is only a finite number of number fields K whose genus does not exceed g_0. The same is true for function fields with a given constant field.

We always doubt, whether the proper definition of genus in the number field case should be $g = \log\sqrt{|D|}$, which we adopt in this paper, or $g = \log\sqrt{|D|}+1$. The latter has the advantage that, for an unramified extension, $g-1$ is multiplied by the degree of the extension (see also [6]). The former one has the advantage that $\mathbb{Q}$ is of genus 0 and has no unramified extensions, just as a curve of genus 0 should. However, for infinite number fields and other asymptotic considerations this is irrelevant, both definitions giving same results.

We call a sequence $\{K_i\}$ of global fields a *family* if K_i is non-isomorphic to K_j for $i \neq j$. A family is called a *tower* if also $K_i \subset K_{i+1}$ for any i. In any family $g_i \to \infty$ for $i \to \infty$. In the function field case we always assume that the constant field of all K_i is one and the same field $\mathbb{F}_r$.

In the number field case let

$$n_i = [K_i : \mathbb{Q}] = r_1(K_i) + 2r_2(K_i),$$

where r_1 and r_2 stand for the numbers of real and (pairs of) complex embeddings. We suppose also that $g_i > 0$ for any i , i.e., $K_i \neq \mathbb{Q}$ in the number field case, and K_i is non-isomorphic to $\mathbb{F}_r(T)$ in the function field case, this assumption does not restrict the generality of our considerations.

We consider the set $A = \{\mathbb{R}, \mathbb{C}; 2, 3, 4, 5, 7, 8, 9, \ldots\}$ of all prime powers plus two auxiliary symbols $\mathbb{R}$ and $\mathbb{C}$ as the set of indices. The parameters we are going to present will be indexed by elements $\alpha \in A$. In the function field case over $\mathbb{F}_r$ the set A is reduced to $A_r = \{r, r^2, r^3, \ldots\}$, meaning that for $\alpha \in A \setminus A_r$ the parameters, we are looking at, vanish.

For a prime power q we set

$$N_q(K_i) := |\{v \in P(K_i) : \mathrm{Norm}(v) = q\}| \ ,$$

where $P(K_i)$ is the set of non-archimedean places of K_i. We also put $N_{\mathbb{R}}(K_i) = r_1(K_i)$ and $N_{\mathbb{C}}(K_i) = r_2(K_i)$.

By h_i we denote the class-number of K_i (which equals the number of $\mathbb{F}_r$-rational points on the Jacobian of K_i in the function field case); R_i denotes the regulator of K_i in the number field case and equals 1 in the function field case.

For an infinite global field $\mathcal{K} = \bigcup K_i$ and $\alpha \in A$ let us introduce the following quantities:

$$\phi_\alpha = \phi_\alpha(\mathcal{K}) := \lim_{i \to \infty} \frac{N_\alpha(K_i)}{g_i}.$$

Of course, we need to prove that these limits exist and do not depend on the tower.

Note that $\phi_{\mathbb{R}}$ and $\phi_{\mathbb{C}}$ are finite, since the ratio $n/g_i = r_1(K_i)/g_i + 2r_2(K_i)/g_i$ is bounded on the set of all number fields $\neq \mathbb{Q}$ by the Minkowski bound.

More generally, we call a family $\mathcal{K} = \{K_i\}$, $i = 1, 2, \ldots$, of global fields *asymptotically exact* if and only if for any $\alpha \in A$ there exist the limit

$$\phi_\alpha = \phi_\alpha(\mathcal{K}) := \lim_{i \to \infty} \frac{N_\alpha(K_i)}{g_i}.$$

We call the family $\mathcal{K}$ *asymptotically good* (respectively, *bad*) if there exists $\alpha \in A$ with $\phi_\alpha > 0$ (respectively, $\phi_\alpha = 0$ for any $\alpha \in A$).

It is important to point out that, by abuse of notation, $\mathcal{K}$ is used both for an infinite global field and for an asymptotically exact family. This is reasonable since below we prove that for an infinite global field all our definitions and results do not depend on the choice of the tower.

The notion of an asymptotically exact family is much more general than that of a tower. In particular, a simple diagonal argument shows that any family contains an asymptotically exact subfamily.

The quantities $\phi = \{\phi_\alpha\}$ give rise to the following definition. The *limit zeta-function* of an asymptotically exact family is defined by the product

$$\zeta_{\mathcal{K}}(s) = \zeta_\phi(s) = \prod_q (1 - q^{-s})^{-\phi_q} ,$$

q running over all prime powers. Here and below, by raising to a complex power a function in s defined for $\operatorname{Re} s > a \geq 0$ and such that its values are real positive for real $s > a$, we mean unique analytic continuation of what is real positive for real s. The *"completed" zeta-function* is defined in the number field case as

$$\tilde{\zeta}_{\mathcal{K}}(s) = \tilde{\zeta}_\phi(s) = e^s 2^{-\phi_{\mathbb{R}}} \pi^{-s\phi_{\mathbb{R}}/2} (2\pi)^{-s\phi_{\mathbb{C}}} \Gamma(\frac{s}{2})^{\phi_{\mathbb{R}}} \Gamma(s)^{\phi_{\mathbb{C}}} \prod_q (1 - q^{-s})^{-\phi_q}.$$

In the function field case, consistent with our convention, we set

$$\tilde{\zeta}_{\mathcal{K}}(s) = \tilde{\zeta}_\phi(s) = r^s \prod_{m=1}^{\infty} (1 - r^{-ms})^{-\phi_{r^m}}.$$

The product defining zeta-functions $\zeta_\phi(s)$ and $\tilde{\zeta}_\phi(s)$ absolutely converges for $\operatorname{Re}(s) \geq 1$. These functions depend only on $\phi = \{\phi_\alpha\}$ and do not depend on the particular sequence of global fields. Therefore, we have defined $\zeta_{\mathcal{K}}(s)$ and $\tilde{\zeta}_{\mathcal{K}}(s)$ for any infinite global field $\mathcal{K}$.

The zeta-function of a family is thus the limit of g-th roots of usual zeta-functions of its fields K_i. Moreover, for $\operatorname{Re} s \geq 1 + \varepsilon$ the convergence is uniform.

It is also true that a family is asymptotically exact if and only if the limit $\lim \zeta_{K_i}(s)^{1/g_i}$ exists.

Let now in the number field case

$$\xi_{\mathcal{K}}(s) = \xi_\phi(s) = (\log \tilde{\zeta}_\phi)' = \tilde{\zeta}'_\phi / \tilde{\zeta}_\phi =$$

$$1 - \frac{\phi_{\mathbb{R}}}{2} \log \pi - \phi_{\mathbb{C}} \log 2\pi + \frac{1}{2} \phi_{\mathbb{R}} \psi(\frac{s}{2}) + \phi_{\mathbb{C}} \psi(s) - \sum_q \phi_q \frac{\log q}{q^s - 1},$$

where $\psi(s) = \frac{\Gamma'}{\Gamma}(s)$, and in the function field case let

$$\xi_{\mathcal{K}}(s) = \xi_\phi(s) = (\log_r \tilde{\zeta}_\phi)' = 1 - \sum_{m=1}^{\infty} \frac{m\phi_{r^m}}{r^{ms} - 1}.$$

Studying the number field case we often assume the generalized Riemann hypothesis (GRH) to hold for number fields in question, but the most part of our results also has an unconditional (weaker) formulation. To distinguish between the two, we always write GRH in relevant cases. Note that the function field case does not need it, GRH being proved.

Under GRH the above products converge absolutely for $\operatorname{Re}(s) \geq \frac{1}{2}$, and we have the following

GRH Theorem A (GRH Basic Inequality). *For an infinite global field $\mathcal{K}$ (and for any asymptotically exact family of global fields)*

$$\xi_{\mathcal{K}}(\frac{1}{2}) \geq 0.$$

This theorem imposes severe restrictions on the possible values of $\phi = \{\phi_\alpha\}$, namely

GRH Corollary A (GRH Basic Inequality). *For an infinite global field (and for any asymptotically exact family of global fields)*

$$\sum_q \frac{\phi_q \log q}{\sqrt{q}-1} + \phi_{\mathbb{R}}(\log 2\sqrt{2\pi} + \frac{\pi}{4} + \frac{\gamma}{2}) + \phi_{\mathbb{C}}(\log 8\pi + \gamma) \leq 1,$$

the sum being taken over all prime powers q.

In the number field case this result generalizes the GRH Odlyzko–Serre inequality on discriminants of number fields. Indeed, all terms being non-negative, we obtain

$$\phi_{\mathbb{R}}(\log 2\sqrt{2\pi} + \frac{\pi}{4} + \frac{\gamma}{2}) + \phi_{\mathbb{C}}(\log 8\pi + \gamma) \leq 1,$$

which means

$$D \geq (8\pi e^{\gamma + \frac{\pi}{2}})^{r_1} (8\pi e^{\gamma})^{2r_2} e^{o(n)}.$$

In the function field case over $\mathbb{F}_r$ the inequality simplifies. (Recall our convention that log, meaning $\log_e$ in the number field case, means $\log_r$ in the function field case.)

Corollary A′ (Basic Inequality in the Function Field Case). *For an infinite function field (and for any asymptotically exact family of function fields)*

$$\sum_{m=1}^{\infty} \frac{m\phi_{r^m}}{r^{m/2}-1} \leq 1.$$

This result generalizes the Drinfeld–Vlăduţ theorem, saying that for the number N of points of degree one on algebraic curves over a finite field $\mathbb{F}_r$ we have

$$N \leq (\sqrt{r}-1)g + o(g),$$

as g tends to infinity, i.e., that $\phi_r \leq \sqrt{r}-1$. Indeed, it is enough to omit in the sum all terms except the first one.

Contemplating the statements of Theorem A and Corollary A one gets interested in the value of $\xi_{\mathcal{K}}(\frac{1}{2})$ which equals the *deficiency*, the difference between the right hand side and the left hand side of the inequality of Corollary A. This deficiency happens to be related to the limit distribution of zeroes of zeta-functions.

We suppose again GRH to hold. Let $\mathcal{K} = \{K_j\}$ be an asymptotically exact family of number fields. For each K_j we define the measure on $\mathbb{R}$

$$\Delta_{K_j} := \frac{\pi}{g_{K_j}} \sum_{\zeta_{K_j}(\rho)=0} \delta_{t(\rho)},$$

where $t(\rho) = (\rho - \frac{1}{2})/i$, and ρ runs over all non-trivial zeroes of the zeta-function $\zeta_{K_j}(s)$. Because of GRH $t(\rho)$ is real, and Δ_{K_j} is a discrete measure on $\mathbb{R}$. Moreover, Δ_{K_j} is a measure of slow growth.

GRH Theorem B (GRH Explicit Formula). *For an infinite number field $\mathcal{K}$ (and for any asymptotically exact family of number fields) there exists the limit*

$$\Delta_{\mathcal{K}} = \lim_{j\to\infty} \Delta_{K_j}$$

in the space of measures of slow growth on $\mathbb{R}$. Moreover, the measure $\Delta_{\mathcal{K}}$ has a continuous density $M_{\mathcal{K}}$,

$$M_{\mathcal{K}}(t) = \mathrm{Re}\left(\xi_{\mathcal{K}}\left(\frac{1}{2}+it\right)\right) =$$

$$1 - \sum_q \phi_q h_q(t)\log q +$$

$$\frac{1}{2}\phi_{\mathbb{R}}\,\mathrm{Re}\,\psi\left(\frac{1}{4}+\frac{it}{2}\right) + \phi_{\mathbb{C}}\,\mathrm{Re}\,\psi\left(\frac{1}{2}+it\right) - \frac{\phi_{\mathbb{R}}}{2}\log\pi - \phi_{\mathbb{C}}\log 2\pi,$$

where

$$h_q(t) = \frac{\sqrt{q}\cos(t\log q) - 1}{q+1-2\sqrt{q}\cos(t\log q)}, \quad \psi(s) = \frac{\Gamma'}{\Gamma}(s).$$

GRH Corollary B (GRH Basic Equality). *For an infinite number field $\mathcal{K}$ (and for any asymptotically exact family of number fields)*

$$\xi_{\mathcal{K}}(\frac{1}{2}) = M_{\mathcal{K}}(0),$$

i.e.,

$$\sum_q \frac{\phi_q\log q}{\sqrt{q}-1} + \phi_{\mathbb{R}}(\log 2\sqrt{2\pi} + \frac{\pi}{4} + \frac{\gamma}{2}) + \phi_{\mathbb{C}}(\log 8\pi + \gamma) = 1 - M_{\mathcal{K}}(0).$$

This means that the difference between 1 and the left hand side of the Basic Inequality, called the deficiency of the infinite global field (or of the family), is in fact the "relative number" of zeroes accumulating at the real critical point $\frac{1}{2}$.

In the function field case the same is true and much easier to prove (cf.[36]). Zeta-functions being periodic, we can of course consider the space of periodic measures on $\mathbb{R}$ to obtain Theorem B and Corollary B in this case. We can also make the formulation simpler using measures on the circle. We normalise the circle to be $\mathbb{R}/2\pi\mathbb{Z}$ represented by $(-\pi, \pi]$. For a zero ρ of the zeta-function $\zeta_{K_j}(s)$ let $t(\rho)$ be defined by

$$t(\rho) = \frac{\rho - \frac{1}{2}}{i} \pmod{2\pi}.$$

Let

$$\Delta_j := \frac{\pi}{g_j} \sum_{\zeta_{K_j}(\rho)=0} \delta_{t(\rho)},$$

where $\delta_{t(\rho)}$ is, as usual, the Dirac measure supported at $t(\rho)$. Then Δ_j is a measure of total mass 2π on $\mathbb{R}/2\pi\mathbb{Z}$, and Δ_j is symmetric with respect to $t \mapsto -t$.

Corollary B′ (Explicit Formula and Basic Equality in the Function Field Case). *In the function field case in the weak topology on the space of measures on $\mathbb{R}/2\pi\mathbb{Z}$ the limit*

$$\Delta_{\mathcal{K}} = \lim_{j\to\infty} \Delta_j$$

exists. Moreover, the measure $\Delta_{\mathcal{K}}$ has a continuous density $M_{\mathcal{K}}$,

$$M_{\mathcal{K}}(t) = \mathrm{Re}(\xi_{\mathcal{K}}(\frac{1}{2} + \frac{i}{\log_e r} t)) = 1 - \sum_{m=1}^{\infty} m\phi_{r^m} h_m(t)$$

for

$$h_m(t) = \frac{r^{m/2}\cos(mt) - 1}{r^m + 1 - 2r^{m/2}\cos(mt)}\ ,$$

which depends only on the family of numbers $\phi = \{\phi_{r^m}\}$ and we have the following Basic Equality:

$$\xi_{\mathcal{K}}(\frac{1}{2}) = 1 - \sum_{m=1}^{\infty} \frac{m\phi_{r^m}}{r^{m/2} - 1} = M_{\mathcal{K}}(0).$$

In the number field case with no GRH at hand, the results become considerably weaker.

Theorem C (Unconditional Basic Inequality). *For an infinite number field $\mathcal{K}$ (and for any asymptotically exact family of number fields)*

$$\xi_{\mathcal{K}}(1) \geq 0.$$

Corollary C (Unconditional Basic Inequality). *For an infinite number field (and for any asymptotically exact family of number fields)*

$$\sum_q \frac{\phi_q \log q}{q-1} + (\gamma/2 + \log 2\sqrt{\pi})\phi_{\mathbb{R}} + (\gamma + \log 2\pi)\phi_{\mathbb{C}} \leq 1.$$

This time, omitting all non-archimedean terms, we get the unconditional Stark inequality,

$$D \geq (4\pi e^{\gamma})^{r_1}(2\pi e^{\gamma})^{2r_2} e^{o(n)}.$$

The unconditional Odlyzko inequality

$$D \geq (4\pi e^{\gamma+1})^{r_1}(4\pi e^{\gamma})^{2r_2} e^{o(n)}$$

can also be generalized, using non-archimedean places.

Our next result concerning zeta-functions concerns the behaviour of class numbers and regulators. For an asymptotically exact family $\mathcal{K}$ of global fields we would like to consider the limit

$$\mathrm{BS}(\mathcal{K}) = \lim_{i\to\infty} \frac{\log h_i R_i}{g_i}.$$

Under certain conditions, as we shall explain below, this limit exists and depends only on the set of numbers $\phi = \{\phi_\alpha\}$. Therefore, $\mathrm{BS}(\mathcal{K})$ is well defined for an infinite global field $\mathcal{K}$, as well as for any asymptotically exact family $\mathcal{K}$. We can also define

$$\varkappa(\mathcal{K}) = \lim_{i\to\infty} \frac{\log \varkappa_i}{g_i},$$

$\varkappa_i$ being the residue of $\zeta_{K_i}(s)$ at 1; this invariant exists under the same conditions.

The value of $\mathrm{BS}(\mathcal{K})$ is described by the Brauer–Siegel theorem. In our terms the classical Brauer–Siegel theorem states:

We have

$$\mathrm{BS}(\mathcal{K}) = 1 \textit{ and } \varkappa(\mathcal{K}) = 0,$$

if the family $\mathcal{K}$ satisfies the following two conditions:

(i) the family $\mathcal{K}$ is asymptotically bad;

(ii) either GRH holds, or all fields K_i are normal over $\mathbb{Q}$.

Indeed, the assumption $n/\log\sqrt{|D|} \to 0$, usually used in the statement, means $\phi_\alpha = 0$ for all α, since it follows that $\phi_\mathbb{R} = \phi_\mathbb{C} = 0$ and for a prime p one has

$$\sum_{m=1}^{\infty} m\phi_{p^m} \le \phi_\mathbb{R} + 2\phi_\mathbb{C}.$$

We are going to generalize the Brauer–Siegel theorem disposing of the first condition.

GRH Theorem D (GRH Generalized Brauer–Siegel Theorem). *For an infinite global field $\mathcal{K}$ (and for any asymptotically exact family of global fields) the limits $\mathrm{BS}(\mathcal{K})$ and $\varkappa(\mathcal{K})$ exist and we have*

$$\mathrm{BS}(\mathcal{K}) = \log \tilde{\zeta}_\mathcal{K}(1),$$

$$\varkappa(\mathcal{K}) = \log \zeta_\mathcal{K}(1).$$

GRH Corollary D (GRH Generalized Brauer–Siegel Theorem). *For an infinite global field (and for any asymptotically exact family of global fields)*

$$\mathrm{BS}(\mathcal{K}) = 1 + \sum_q \phi_q \log\frac{q}{q-1} - \phi_\mathbb{R}\log 2 - \phi_\mathbb{C}\log 2\pi,$$

$$\varkappa(\mathcal{K}) = \sum_q \phi_q \log\frac{q}{q-1},$$

the sum being taken over all prime powers q.

In the function field case, of course,

$$\mathrm{BS}(\mathcal{K}) = \lim_{i\to\infty} \frac{\log_r h_i}{g_i},$$

where h_i is the number of $\mathbb{F}_r$-points on the Jacobian of the curve X_i corresponding to the field K_i. The other parameter $\varkappa(\mathcal{K}) = \mathrm{BS}(\mathcal{K}) - 1$, and thus becomes uninteresting.

Corollary D′ (Generalized Brauer–Siegel Theorem in the Function Field Case). *For an infinite function field $\mathcal{K}$ (and for any asymptotically exact family of function fields) the limit $\mathrm{BS}(\mathcal{K})$ exists and we have*

$$\mathrm{BS}(\mathcal{K}) = 1 + \sum_{m=1}^{\infty} \phi_{r^m} \log_r \frac{r^m}{r^m - 1}.$$

In the number field case half of Theorem D does not depend on GRH, namely we prove

Theorem E (Generalized Brauer–Siegel Inequality). *For an infinite number field (and for any asymptotically exact family of number fields)*

$$\limsup_{i\to\infty} \frac{\log(h_i R_i)}{g_i} \le 1 + \sum_q \phi_q \log \frac{q}{q-1} - \phi_{\mathbb{R}} \log 2 - \phi_{\mathbb{C}} \log 2\pi,$$

$$\limsup_{i\to\infty} \frac{\log(\varkappa_i)}{g_i} \le \sum_q \phi_q \log \frac{q}{q-1},$$

the sum being taken over all prime powers q.

As yet, we are unable to prove the generalized Brauer–Siegel theorem unconditionally. In the general case, even the classical Brauer–Siegel theorem is not known, one needs normality of the fields in question. However, for an infinite number field with an auxiliary condition this becomes possible.

Theorem F (Unconditional Generalized Brauer–Siegel Theorem for Infinite Number Fields). *For an infinite almost normal asymptotically good number field $\mathcal{K}$ the limits $\mathrm{BS}(\mathcal{K})$ and $\varkappa(\mathcal{K})$ exist and we have*

$$\mathrm{BS}(\mathcal{K}) = \log \tilde{\zeta}_{\mathcal{K}}(1),$$

$$\varkappa(\mathcal{K}) = \log \zeta_{\mathcal{K}}(1).$$

Corollary F (Unconditional Generalized Brauer–Siegel Theorem for Infinite Number Fields). *For an infinite almost normal asymptotically good number field $\mathcal{K}$*

$$\mathrm{BS}(\mathcal{K}) = 1 + \sum_q \phi_q \log \frac{q}{q-1} - \phi_{\mathbb{R}} \log 2 - \phi_{\mathbb{C}} \log 2\pi,$$

$$\varkappa(\mathcal{K}) = \sum_q \phi_q \log \frac{q}{q-1},$$

the sum being taken over all prime powers q.

Note that the unconditional classical Brauer–Siegel theorem for normal fields *does not* follow from our results.

Next question is that of the possible asymptotic behaviour of the Brauer–Siegel ratios $\log(hR)/g$ and $\log(\varkappa)/g$. We prove

GRH Theorem G (GRH Bounds). *For any family of number fields*

$$\mathrm{BS}_{\mathrm{lower}} \le \liminf_{i\to\infty} \frac{\log(h_i R_i)}{g_i} \le \limsup_{i\to\infty} \frac{\log(h_i R_i)}{g_i} \le \mathrm{BS}_{\mathrm{upper}},$$

$$0 \le \liminf_{i\to\infty} \frac{\log(\varkappa_i)}{g_i} \le \limsup_{i\to\infty} \frac{\log(\varkappa_i)}{g_i} \le \varkappa_{\mathrm{upper}},$$

where

$$\mathrm{BS}_{\mathrm{lower}} = 1 - \frac{\log 2\pi}{\gamma + \log 8\pi} \approx 0.5165...,$$

$$\mathrm{BS}_{\mathrm{upper}} = 1 + \frac{\log\frac{3}{2} + \log\frac{5}{4} + \log\frac{7}{6}}{\frac{\gamma}{2} + \frac{\pi}{4} + \log 2\sqrt{2\pi} + \frac{\log 2}{\sqrt{2}-1} + \frac{\log 3}{\sqrt{3}-1} + \frac{\log 5}{\sqrt{5}-1} + \frac{\log 7}{\sqrt{7}-1}} \approx 1.0938...,$$

$$\varkappa_{\mathrm{upper}} = \frac{\log 2 + \log\frac{3}{2}}{\frac{\gamma}{2} + \log 2\sqrt{2\pi} + \frac{\log 2}{\sqrt{2}-1} + \frac{\log 3}{\sqrt{3}-1}} \approx 0.2164\ldots$$

In what follows we also give some bounds specific for the totally real case and for the totally complex one.

The function field case was treated in our paper [36]. In our terms we have

Theorem G′ (Function Field Bounds). *For any family of function fields over $\mathbb{F}_r$*

$$1 \le \liminf_{i\to\infty} \frac{\log_r h_i}{g_i} \le \limsup_{i\to\infty} \frac{\log_r h_i}{g_i} \le 1 + (\sqrt{r}-1)\log_r \frac{r}{r-1}.$$

In the number field case, as usual, without GRH Theorem G weakens.

Theorem H (Unconditional Bounds). *For any family of number fields*

$$\limsup_{i\to\infty} \frac{\log(h_i R_i)}{g_i} \le \mathrm{BS}_{\mathrm{unc,upper}},$$

$$\limsup_{i\to\infty} \frac{\log(\varkappa_i)}{g_i} \le \varkappa_{\mathrm{unc,upper}},$$

where

$$\mathrm{BS}_{\mathrm{unc,upper}} = 1 + \frac{\sum\limits_{\substack{p=3\\ prime}}^{23} \log\frac{p}{p-1}}{\frac{\gamma}{2} + \frac{1}{2} + \log 2\sqrt{\pi} + 2\sum\limits_{\substack{p=2\\ prime}}^{23} \log p \sum\limits_{m=1}^{\infty} \frac{1}{p^m+1}} \approx 1.1588\ldots,$$

$$\varkappa_{\mathrm{unc,upper}} = 1 + \frac{\sum\limits_{\substack{p=3\\ prime}}^{5} \log \frac{p}{p-1}}{\frac{\gamma}{2} + \log 2\sqrt{\pi} + 2 \sum\limits_{\substack{p=2\\ prime}}^{5} \log p \sum\limits_{m=1}^{\infty} \frac{1}{p^m+1}} \approx 0.3151\ldots$$

For an infinite almost normal asymptotically good number field $\mathcal{K}$ we also have the lower bound

$$\liminf_{i\to\infty} \frac{\log(h_i R_i)}{g_i} \geq \mathrm{BS}_{\mathrm{unc,lower}},$$

where

$$\mathrm{BS}_{\mathrm{unc,lower}} = 1 - \frac{\log 2\pi}{\gamma + \log 4\pi} \approx 0.4087\ldots$$

Knowing that GRH–possible values of the Brauer–Siegel ratio lie in the interval

$$(0.5165..., 1.0938...),$$

and having in mind the classical value 1 of the Brauer–Siegel theorem itself, we are curious to know whether in our more general setting there exist examples when it differs from 1.

They do exist. The method to construct such examples of infinite number fields is to take the limit of a class field tower given by some splitting conditions. In particular, we get

GRH Theorem I. *The field*

$$K = \mathbb{Q}(\cos\frac{2\pi}{11}, \sqrt{2}, \sqrt{-23})$$

has an infinite unramified 2*-tower* $\mathcal{K}$*, for which* $\mathrm{BS}(\mathcal{K}) \in (\mathrm{BS}_{\mathrm{lower}}(\mathcal{K}), \mathrm{BS}_{\mathrm{upper}}(\mathcal{K}))$*, where*

$$\mathrm{BS}_{\mathrm{lower}}(\mathcal{K}) = 1 - \frac{10\log 2\pi}{g},$$

$$\mathrm{BS}_{\mathrm{upper}}(\mathcal{K}) = \mathrm{BS}_{\mathrm{lower}}(\mathcal{K}) + \frac{(\sqrt{23}-1)\log\frac{23}{22}}{\log 23}\left(1 - \frac{10(\gamma + \log 8\pi)}{g}\right),$$

i.e., approximately

$$0.5939\ldots \leq \mathrm{BS}(\mathcal{K}) \leq 0.6025\ldots.$$

Note that we do not need examples giving lower bounds for $\varkappa(\mathcal{K})$ since any asymptotically bad infinite number field, for example any tower of fields abelian over $\mathbb{Q}$, attains the obvious lower bound $\varkappa(\mathcal{K}) = 0$.

Without GRH the upper bound is less precise.

Theorem J. *The field*

$$K = \mathbb{Q}(\cos\frac{2\pi}{11}, \sqrt{2}, \sqrt{-23})$$

has an infinite unramified 2*-tower* $\mathcal{K}$*, for which* $\mathrm{BS}(\mathcal{K}) \in (\mathrm{BS}_{\mathrm{lower}}(\mathcal{K}), \mathrm{BS}_{\mathrm{unc,upper}}(\mathcal{K}))$*, where* $\mathrm{BS}_{\mathrm{lower}}(\mathcal{K})$ *is as above and* $\mathrm{BS}_{\mathrm{unc,upper}}(\mathcal{K}) \approx 0.7108\ldots$*.*

The upper bound we have got shows that the condition $n/\log|D| \to 0$ (or in our terms $\phi_\alpha = 0$ for every α) in the classical Brauer–Siegel theorem is indeed indispensable. In other words, the Brauer–Siegel ratio $\mathrm{BS}(\mathcal{K})$ can be strictly less than 1. Can it also be strictly greater than 1? Can $\varkappa(\mathcal{K})$ be strictly positive? Here is an example.

GRH Theorem K. *The field*

$$K = \mathbb{Q}(\sqrt{11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47\cdot 53\cdot 59\cdot 61\cdot 67})$$

has an infinite unramified 2*-tower* $\mathcal{K}$ *in which nine prime ideals lying over* 2*,* 3*,* 5*,* 7 *and* 71 *split completely. Then* $\mathrm{BS}(\mathcal{K}) \in (\mathrm{BS}_{\mathrm{lower}}(\mathcal{K}), \mathrm{BS}_{\mathrm{upper}}(\mathcal{K}))$ *and* $\varkappa(\mathcal{K}) \in (\varkappa_{\mathrm{lower}}(\mathcal{K}), \varkappa_{\mathrm{upper}}(\mathcal{K}))$*, where*

$$\mathrm{BS}_{\mathrm{lower}}(\mathcal{K}) = 1 + \frac{2\log\frac{3}{2} + 2\log\frac{5}{4} + 2\log\frac{7}{6} + \log\frac{5041}{5040}}{g},$$

$$\mathrm{BS}_{\mathrm{upper}}(\mathcal{K}) = \mathrm{BS}_{\mathrm{lower}}(\mathcal{K}) + \frac{1}{g}\sum_{p=11}^{47}\log\frac{p}{p-1} +$$

$$\frac{\sqrt{53}-1}{g\log 53}\left(g - \gamma - \frac{\pi}{2} - \log 8\pi - 2\sum_{p=2}^{7}\frac{\log p}{\sqrt{p}-1} - \frac{\log 71^2}{70} - \sum_{p=11}^{47}\frac{\log p}{\sqrt{p}-1}\right)\log\frac{53}{52},$$

$$\varkappa_{\mathrm{lower}}(\mathcal{K}) = \frac{2\log 2 + 2\log\frac{3}{2} + 2\log\frac{5}{4} + 2\log\frac{7}{6} + \log\frac{5041}{5040}}{g},$$

$$\varkappa_{\mathrm{upper}}(\mathcal{K}) = \mathrm{BS}_{\mathrm{upper}}(\mathcal{K}) - 1 + \frac{2\log 2}{g},$$

the sums being taken over prime p*'s. Numerically*

$$\mathrm{BS}(\mathcal{K}) \in (1.0602\ldots, 1.0798\ldots),$$

$$\varkappa(\mathcal{K}) \in (0.1135\ldots, 0.1331\ldots).$$

Here, as well, without GRH the upper bound changes.

Theorem L. *The field*

$$K = \mathbb{Q}(\sqrt{11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47\cdot 53\cdot 59\cdot 61\cdot 67})$$

has an infinite unramified 2*-tower* $\mathcal{K}$ *in which nine prime ideals lying over* 2*,* 3*,* 5*,* 7 *and* 71 *split completely. Then* $\mathrm{BS}(\mathcal{K}) \in (\mathrm{BS}_{\mathrm{lower}}(\mathcal{K}), \mathrm{BS}_{\mathrm{unc,upper}}(\mathcal{K}))$ *and* $\varkappa(\mathcal{K}) \in (\varkappa_{\mathrm{lower}}(\mathcal{K}), \varkappa_{\mathrm{unc,upper}}(\mathcal{K}))$*, where* $\mathrm{BS}_{\mathrm{lower}}(\mathcal{K})$ *and* $\varkappa_{\mathrm{lower}}(\mathcal{K})$ *are as above,* $\mathrm{BS}_{\mathrm{unc,upper}}(\mathcal{K}) \approx 1.0951\ldots$*,* $\varkappa_{\mathrm{unc,upper}}(\mathcal{K}) \approx 0.1454\ldots$*.*

The values of different bounds for $\mathrm{BS}(\mathcal{K})$ developed in this paper form the following table.

		lower bound	lower example	upper example	upper bound
GRH	all fields	0.5165	0.5939–0.6025	1.0602–1.0798	1.0938
	totally real	0.7419	0.8009–0.8648	1.0602–1.0798	1.0938
	totally complex	0.5165	0.5939–0.6025	1.0482–1.0653	1.0764
unconditional	all fields	0.4087	0.5939–0.7108	1.0602–1.0921	1.1588
	totally real	0.6625	0.8009–0.9248	1.0602–1.0921	1.1588
	totally complex	0.4087	0.5939–0.7108	1.0482–1.0951	1.0965

And here is the table for $\varkappa(\mathcal{K})$. Note that the lower bound $\varkappa(\mathcal{K}) = 0$ is always attainable.

		upper example	upper bound
GRH	all fields	0.1135–0.1331	0.2164
	totally real	0.1135–0.1331	0.1874
	totally complex	0.1162–0.1333	0.2164
unconditional	all fields	0.1135–0.1454	0.3151
	totally real	0.1135–0.1454	0.2816
	totally complex	0.1162–0.1631	0.3151

In the function field case an example of $\mathcal{K}$ with $\mathrm{BS}(\mathcal{K}) = 1$, $\varkappa(\mathcal{K}) = 0$ is provided by any tower with $\phi_\alpha = 0$ for every α. In particular, any tower of fields abelian over $\mathbb{F}_r(t)$ has this property. An example reaching the upper bound must have $\phi_r = \sqrt{r} - 1$ and $\phi_\alpha = 0$ for every other α. The existence of such towers is known only when r is a square. For a square r, different modular towers enjoy this property.

As an application of the Generalized Brauer-Siegel Theorem one obtains a lower bound for regulators of number fields in asymptotically good families which is better than Zimmert's bound.

Theorem M (Regulator Bound). *For an asymptotically good tower of number fields $\mathcal{K} = \{K_i\}$ we have*

$$\liminf_{i\to\infty} \frac{\log R_i}{g_i} \geq (\log \sqrt{\pi e} + \frac{\gamma}{2})\phi_{\mathbb{R}} + (\log 2 + \gamma)\phi_{\mathbb{C}}.$$

Under GRH we get the same estimate for any asymptotically good family of number fields.

Our work resulting in this paper was started about ten years ago. Now we are convinced that there is a non-trivial theory of infinite global fields, though we do not yet understand what it should really look like.

The paper starts with generalities on infinite global fields and their zeta-functions. In Section 2 we introduce the invariants $\phi_\alpha(\mathcal{K})$. In Section 3 we prove the first form of the Basic Inequality. Then we introduce zeta-functions

(Section 4) and prove the Explicit Formula (Section 5). In Section 6 we discuss possible directions of further study of infinite global fields.

Part 2 is consacrated to the Brauer–Siegel theorem. We prove the generalized Brauer–Siegel theorem in Section 7, as well as regulator bounds. In Section 8 we provide the bounds for the Brauer–Siegel ratio. Section 9 is devoted to class field towers. We finish by discussing open questions.

It is a pleasure for us to acknowledge the previous work without which this paper would have never been written. Any unified treatment of number and function fields makes appeal to the heritage of A.Weil. A great part of this work develops two classical results, the Odlyzko–Serre inequalities and the Brauer–Siegel theorem, both ideologically and technically. We first understood what is going on in the function field case [36]. Y. Ihara [12] obtained most part of results of Section 3 below in the particular case of unramified towers, both in the function field case and in the number field one. (Unfortunately, we were unaware of [12] while writing [36].) His technique helped us a lot. A version of a particular case (the asymptotically bad one) of GRH Theorem B is the main result of Lang's paper [15]. Discussions with many of our friends and collegues were extremely useful. We would especially like to thank for many valuable remarks J.-P.Serre and the anonymous author of a 13 page long referee report on one of the previous versions of this paper. We thank G.Lachaud for his interest in our work and for attracting our attention to the question about the minimum zeta-zero.

It is our greatest pleasure to devote this paper to our teacher Yuri Ivanovich Manin, who taught us to consider number fields, zeta functions and algebraic curves as different facets of one diamond. Congratulating him with his 65th birthday, we wish him many happy returns of the day.

Part I

Zeta-function of an infinite global field

Let us repeat the definition. An *infinite global field* $\mathcal{K}$ is either an infinite algebraic extension of $\mathbb{Q}$, or an infinite algebraic extension of $\mathbf{Q}_r = \mathbb{F}_r(t)$ such that $\mathcal{K} \cap \bar{\mathbb{F}}_r = \mathbb{F}_r$. In the first case we call $\mathcal{K}$ an *infinite number field*, in the second an *infinite function field over* $\mathbb{F}_r$.

Our main problem here is to find out parameters of infinite global fields and to construct a zeta-function of such a field.

2 Invariants of infinite global fields

Here we give some basics on infinite global fields and asymptotically exact families showing that these notions are worth studying.

Lemma 2.1. *For any given g_0 there is only a finite number of number fields K whose genus does not exceed g_0. The same is true for function fields over a given constant field (considered up to an isomorphism).*

Proof. In the number field case this is proved by the geometry of numbers (cf. [16], Theorem V.4.5). In the function field case this follows from the existence of moduli spaces for genus g curves. Indeed, those are varieties over the ground field which is finite, and thus they have but a finite number of points defined over it. $\square$

For a global field K and for a prime power q we set

$$N_q(K) := |\{v \in P(K) : \mathrm{Norm}\, v = q\}|,$$

where $P(K)$ is the set of non-archimedean places of K. We also write $N_{\mathbb{R}}(K) = r_1(K)$ for the number of real places and $N_{\mathbb{C}}(K) = r_2(K)$ for that of complex ones. In the function field case we set $N_{\mathbb{R}}(K) = N_{\mathbb{C}}(K) = 0$.

We call a sequence $\{K_i\}$ of global fields a *family* if K_i is non-isomorphic to K_j for $i \neq j$. In any family $g_i = g(K_i) \to \infty$ for $i \to \infty$. (In the function field case we always assume that the constant field of all K_i equals one and the same $\mathbb{F}_r$.)

Definition 2.1. We call a family $\{K_i\}$ *asymptotically exact* if and only if for any $\alpha \in A = \{\mathbb{R}, \mathbb{C}; 2, 3, 4, 5, 7, 8, 9, \ldots\}$ there exists a limit

$$\phi_\alpha := \lim_{i\to\infty} \frac{N_\alpha(K_i)}{g_i}.$$

Note that we can as well divide by $g_i - 1$ instead of g_i, since for almost all i we have $g_i > 1$ and $g_i \to \infty$. We shall use this division in the function field case.

Lemma 2.2. *Every family of global fields $\{K_i\}$ contains an asymptotically exact subfamily.*

Proof. In the number field case, for any q we have $N_q(K_i) \leq n(K_i)$, where $n(K) = [K : \mathbb{Q}]$. On the other hand, since $K_i \neq \mathbb{Q}$ there exists a universal constant C such that $|D_{K_i}|^{1/n} \geq C$. This follows, e.g., from the Odlyzko–Serre inequalities (or even from the Minkowski constant argument), and Lemma 2.1. Thus $n(K_i)/g(K_i) \leq 1/\log C$, and all limit points of $N_q(K_i)/g(K_i)$ lie between 0 and $1/\log C$.

Therefore, for each separate q there is a limit over some subsequence of $\mathbf{K} = \{K_i\}$. The same is true for $r_\alpha(K_i)/g(K_i)$, $\alpha = \mathbb{R}, \mathbb{C}$.

Choose such a subsequence $\mathbf{K}_0$ that $\phi_{\mathbb{R}}$ exists. Take its subsequence $\mathbf{K}_1$ where $\phi_{\mathbb{C}}$ exists, then its subsequence $\mathbf{K}_2$ where ϕ_2 exists, and so on. Now take a diagonal sequence, i.e., such that $K_1 \in \mathbf{K}_1$, $K_2 \in \mathbf{K}_2$, etc. For this sequence ϕ_α exists for any α.

The function field case is treated similarly. $\square$

Remark 2.1. As it was pointed out by J.-P. Serre, this proof only uses the fact that the space of positive measures with mass 1 is a metric compact space.

The lemma shows that the study of any family can be reduced in a certain sense to that of asymptotically exact families, and from now on we mostly suppose all our families to be asymptotically exact.

Lemma 2.3. *Let $L \supseteq K$. Then, in the number field case, for $g(K) \geq 1$,*

$$\frac{N_{\mathbb{R}}(L)}{g(L)} + 2\frac{N_{\mathbb{C}}(L)}{g(L)} \leq \frac{N_{\mathbb{R}}(K)}{g(K)} + 2\frac{N_{\mathbb{C}}(K)}{g(K)},$$

and, for any prime p and any $n \geq 1$,

$$\sum_{m=1}^{n} \frac{mN_{p^m}(L)}{g(L)} \leq \sum_{m=1}^{n} \frac{mN_{p^m}(K)}{g(K)}.$$

In the function field case, for $g(K) \geq 2$ and for any $n \geq 1$,

$$\sum_{m=1}^{n} \frac{mN_{r^m}(L)}{g(L)-1} \leq \sum_{m=1}^{n} \frac{mN_{r^m}(K)}{g(K)-1}.$$

Proof. In the number field case, for an extension $L \supset K$ we have

$$|D_L| \geq |D_K|^{[L:K]},$$

and in the function field case we have

$$g(L) - 1 \geq [L : K](g(K) - 1).$$

On the other hand, if a place v of K is decomposed into a set $\{v_1, v_2, \ldots\}$ of places of L then

$$\prod \operatorname{Norm} v_i \leq (\operatorname{Norm} v)^{[L:K]}.$$

Therefore,

$$\sum_{m=1}^{n} mN_{p^m}(L) \leq [L : K] \sum_{m=1}^{n} mN_{p^m}(K)$$

for any prime p and any $n \geq 1$. Dividing, we get the required monotonicity.

The archimedean inequality and that for function fields are similar. □

Lemma 2.4. *Any infinite tower $K_0 \subset K_1 \subset K_2 \subset \ldots$ is an asymptotically exact family.*

Proof. For a given prime p, by the second inequality of Lemma 2.3, the sequence $\sum\limits_{m=1}^{n} \frac{mN_{p^m}(K_i)}{g(K_i)}, i = 1, 2, \ldots$ is non-increasing for any fixed n, and hence has a limit. (In the function field case we divide by $g(K_i) - 1$.) Taking $n = 1$ we see that ϕ_p exists, setting $n = 2$ we derive the existence of ϕ_{p^2}, etc.

Using the first inequality of Lemma 2.3 we prove the existence of $\phi_{\mathbb{R}}$ and then of $\phi_{\mathbb{C}}$.

The function field case is treated similarly. $\square$

Let $\mathcal{K}$ be an infinite global field. Then $\mathcal{K} = \bigcup_{i=1}^{\infty} K_i$, where $K_1 \subset K_2 \subset K_3 \subset \ldots$, and we can define the corresponding parameters ϕ_α, $\alpha \in A = \{\mathbb{R}, \mathbb{C}; 2, 3, 4, 5, 7, 8, 9, \ldots\}$.

Lemma 2.5. *For an infinite global field $\mathcal{K}$ the parameters ϕ_α do not depend on the choice of the tower $K_1 \subset K_2 \subset K_3 \subset \ldots$.*

Proof. Let $\mathcal{K} = \bigcup_{i=1}^{\infty} K_i = \bigcup_{i=1}^{\infty} L_i$ be two representations. Since each K_i is generated by a finite number of elements of $\mathcal{K}$, it is contained in some L_j. Using the inequalities of Lemma 2.3 we see that the limit of the ratio in question for the tower $\{L_j\}$ is less than or equal to the corresponding value for the tower $\{K_i\}$. As in the proof of Lemma 2.4, we have to treat ϕ_p first, and then to use induction over the degree. We get $\phi_\alpha(\mathcal{L}) \le \phi_\alpha(\mathcal{K})$ and, by symmetry, vice versa. $\square$

Lemma 2.6. *Let $\mathcal{K} \subset \mathcal{L}$ be infinite global fields. Then*

$$\phi_{\mathbb{R}}(\mathcal{K}) + 2\phi_{\mathbb{C}}(\mathcal{K}) \ge \phi_{\mathbb{R}}(\mathcal{L}) + 2\phi_{\mathbb{C}}(\mathcal{L})$$

and for any prime p and any $n \ge 1$

$$\sum_{m=1}^{n} m\phi_{p^m}(\mathcal{K}) \ge \sum_{m=1}^{n} m\phi_{p^m}(\mathcal{L}).$$

In particular,

$$\phi_p(\mathcal{K}) \ge \phi_p(\mathcal{L}).$$

Proof. We follow the same lines as above. Consider a tower $\{L_j\}$ filtering $\mathcal{L}$. Then $\{K_j = \mathcal{K} \cap L_j\}$ filters $\mathcal{K}$. Using the monotonicity of Lemma 2.3 for the pair $K_j \subset L_j$ we get the result. $\square$

Definition 2.2. An infinite global field $\mathcal{K}$ (or an asymptotically exact family $\mathcal{K} = \{K_i\}$) is called *asymptotically bad* if and only if all $\phi_\alpha = 0$. If at least one of the parameters $\phi_\alpha > 0$, it is called *asymptotically good*.

For example, any sequence of global fields of bounded degree is asymptotically bad.

Example 2.1. Families $\{K_i\}$ satisfying the condition

$$n/\log\sqrt{|D|} \to 0$$

of the Brauer–Siegel theorem are asymptotically bad. Indeed, this condition clearly implies $\phi_{\mathbb{R}} = \phi_{\mathbb{C}} = 0$ and then all $\phi_q = 0$ as well, since one has

Lemma 2.7. *For any asymptotically exact family $\mathcal{K} = \{K_i\}$ of number fields and for any prime p one has*

$$\sum_{m=1}^{\infty} m\phi_{p^m} \le \phi_{\mathbb{R}} + 2\phi_{\mathbb{C}}.$$

Proof. Indeed, all places of a number field K whose norm is a power of p lie over p in $\mathbb{Q}$ and the product of their norms is not greater than p^n, where $n = r_1 + 2r_2$ is the degree of K. □

Lemma 2.8. *Let L be a (finite) global field and L^{ab} be its maximal abelian extension. If for an infinite global field $\mathcal{K}$ the field $\mathcal{K} \cap L^{\mathrm{ab}}$ is also infinite, then $\mathcal{K}$ is asymptotically bad. In particular, if $\mathcal{K}$ is abelian over a finite global field, then $\mathcal{K}$ is asymptotically bad.*

In the function field case this lemma is proved in [3]. In the number field case this can be proved using Artin's dicriminant-conductor formula. Since we do not use this result in our paper, we do not prove it here.

Example 2.2. Let now K_1 be a number field with the infinite Hilbert class field tower $\{K_i\}$ — recall that the existence of such fields was proved in [7] — or any other unramified tower. Then the family $\{K_i\}$ is asymptotically good, since the ratio n_i/g_i is constant in such a tower, and thus at least one of $\phi_{\mathbb{R}}$ and $\phi_{\mathbb{C}}$ is nonzero. In fact, in the number field case we do not know essentially different methods for constructing asymptotically good families (however, one can take composita with a fixed number field, one can mix different class field towers, etc).

Remark 2.2. It might happen that $\phi_{\mathbb{R}} + \phi_{\mathbb{C}} > 0$ but $\phi_q = 0$ for any q. Moreover, we suspect this to be the case for the example of Theorem 9.2 below. As yet we are however unable to prove it.

In the function field case there exist three essentially different techniques to construct asymptotically good families: a version of the Golod-Shafarevich method due to Serre [29] which applies to any finite ground field $\mathbb{F}_r$ (cf. [26], [23]), the method of modular curves of different types: classical, Drinfeld, Shimura (cf. [11], [37], [41]) which applies only if r is a square (or sometimes another power) but gives much better results, and explicit construction of the same towers (cf. [5]). For any r one knows the existence of an asymptotically exact family $\mathcal{K}$ with $\phi_r(\mathcal{K}) > 0$ (cf. [29]).

3 Basic inequality

In this section we prove the basic GRH inequality, as well as its weaker versions that do not require GRH. We treat the number field case first.

Let K be a number field of degree n and disriminant D with r_1 real and r_2 pairs of complex embeddings, and let $\zeta_K(s)$ be its Dedekind zeta-function; by $P(K)$ we denote the set of prime divisors of K which can be identified with the set of non-archimedean places of K.

We use the following form of the Guinand-Weil explicit formula (see [20], p.122, eq.2.3).

Let $F(x)$ be a differentiable even real positive function defined on the whole real line $\mathbb{R}$ such that $F(0) = 1$ and there exist positive real constants c and ε such that

$$F(x), F'(x) \leq c e^{-(1/2+\varepsilon)|x|} \text{ as } |x| \to \infty.$$

Define

$$\Phi(s) := \int_{-\infty}^{\infty} F(x) e^{(s-1/2)x} dx \ .$$

Then we have the following formula:

$$\log|D| = r_1 \frac{\pi}{2} + n(\gamma + \log 8\pi) - n \int_0^\infty \frac{1-F(x)}{2\,\mathrm{sh}\,\frac{x}{2}} dx - r_1 \int_0^\infty \frac{1-F(x)}{2\,\mathrm{ch}\,\frac{x}{2}} dx$$

$$-4 \int_0^\infty F(x)\,\mathrm{ch}\,\frac{x}{2} dx + {\sum_\rho}' \Phi(\rho) + 2 \sum_P \sum_{m=1}^{\infty} N(P)^{-m/2} F(m \log N(P)) \log N(P),$$

where in the first sum ρ runs over the zeroes of $\zeta_K(s)$ in the critical strip, $\sum'$ meaning that the ρ and $\bar{\rho}$ terms are to be grouped together, the external sum in the last term is taken over all prime divisors $P \in P(K)$, and in the internal sum $N(P)$ denotes the absolute norm of P.

3.1 GRH basic inequality

Let us now apply this formula to the case of an asymptotically exact family $\mathcal{K} = \{K_i\}$ of number fields.

GRH Theorem 3.1 (GRH Basic Inequality). *For an asymptotically exact family of number fields one has*

$$\sum_q \frac{\phi_q \log q}{\sqrt{q}-1} + \phi_{\mathbb{R}}(\log \sqrt{8\pi} + \frac{\pi}{4} + \frac{\gamma}{2}) + \phi_{\mathbb{C}}(\log 8\pi + \gamma) \leq 1 \ ,$$

the sum being taken over all prime powers q.

(In the special case of unramified towers this theorem was proved by Y.Ihara [12].)

Proof. Let us apply the above explicit formula to the field $K = K_i$ from our sequence, substituting $F(x) = e^{-yx^2}$ for a real positive y. We get

$$\log|D| = r_1 \frac{\pi}{2} + n(\gamma + \log 8\pi) - n \int_0^\infty \frac{1-e^{-yx^2}}{2\,\mathrm{sh}\,\frac{x}{2}} dx$$

$$-r_1 \int_0^\infty \frac{1-e^{-yx^2}}{2\,\mathrm{ch}\,\frac{x}{2}} dx - 4 \int_0^\infty e^{-yx^2}\,\mathrm{ch}\,\frac{x}{2} dx$$

$$+\mathrm{Re}{\sum_t}' \int_{-\infty}^{\infty} e^{itx-yx^2} dx + 2 \sum_P \sum_{m=1}^{\infty} N(P)^{-m/2} e^{-ym^2 \log^2 N(P)} \log N(P),$$

where in the first sum t runs over all reals such that $\zeta_K(1/2+it) = 0$, and the exernal sum in the last term is taken over all prime divisors P of K.

Dividing the last equation by $2g = \log|D|$ and using the relation $n = r_1 + 2r_2$ one gets:

$$1=\frac{r_1}{g}(\frac{\pi}{4}+\frac{\gamma}{2}+\frac{\log 8\pi}{2})+\frac{r_2}{g}(\gamma+\log 8\pi)-\frac{n}{g}\int_0^\infty \frac{1-e^{-yx^2}}{4\operatorname{sh}\frac{x}{2}}dx$$

$$-\frac{r_1}{g}\int_0^\infty \frac{1-e^{-yx^2}}{4\operatorname{ch}\frac{x}{2}}dx-\frac{2}{g}\int_0^\infty e^{-yx^2}\operatorname{ch}\frac{x}{2}dx+\operatorname{Re}\frac{1}{2g}\sum_t\int_{-\infty}^\infty e^{itx-yx^2}dx$$

$$+\sum_q \frac{N(q)}{g}\sum_{m=1}^\infty q^{-m/2}e^{-ym^2\log^2 q}\log q\ ,$$

where $N(q)$ is the number of prime divisors in K of norm q.

Thus $1=T_1+T_2-T_3-T_4-T_5+T_6+T_7$ is presented as a sum of seven terms T_j, $j=1,\ldots,7$. Now we set $y=1/\log g$ and tend g to infinity. Then y tends to zero, and we are going to show that $T_1+T_2+T_7$ tends to the left hand side of the Basic Inequality, while $T_3+T_4+T_5$ tends to zero, T_6 being non-negative, which proves the theorem.

Indeed, it is clear that T_1 tends to $\phi_{\mathbb{R}}(\log 2\sqrt{2\pi}+\frac{\pi}{4}+\frac{\gamma}{2})$, and T_2 tends to $\phi_{\mathbb{C}}(\log 8\pi+\gamma)$. Then, for T_7 we have:

$$T_7=\sum_q \frac{N(q)}{g}\sum_{m=1}^\infty q^{-m/2}e^{-ym^2\log^2 q}\log q\le \sum_q \frac{N(q)}{g}\sum_{m=1}^\infty q^{-m/2}\log q$$

since $e^{-ym^2\log^2 q}\le 1$. On the other hand, since $e^{-ym^2\log^2 q}\ge 1-ym^2\log^2 q$, we get

$$T_7\ge \sum_{q\le\log g}\frac{N(q)}{g}\sum_{m=1}^{[\log^{1/4} g]} q^{-m/2}(1-ym^2\log^2 q)\log q$$

$$\ge (1-\frac{(\log\log g)^2}{\sqrt{\log g}})\sum_{q\le\log g}\frac{N(q)}{g}\sum_{m=1}^{[\log^{1/4} g]} q^{-m/2}\log q.$$

These inequalities show that T_7 tends to

$$\sum_q \phi_q\sum_{m=1}^\infty q^{-m/2}\log q=\sum_q\frac{\phi_q\log q}{\sqrt{q}-1}$$

when $g\to\infty$.

Now let us estimate T_3, T_4, and T_5. Since

$$0\le\int_0^\infty \frac{1-e^{-yx^2}}{4\operatorname{ch}\frac{x}{2}}dx\le\int_0^\infty \frac{1-e^{-yx^2}}{4\operatorname{sh}\frac{x}{2}}dx\ ,$$

if we show that T_3 tends to zero then T_4 also tends to 0. We write

$$\int_0^\infty \frac{1-e^{-yx^2}}{\operatorname{sh}\frac{x}{2}}dx=\int_0^1 \frac{1-e^{-yx^2}}{\operatorname{sh}\frac{x}{2}}dx+\int_1^\delta \frac{1-e^{-yx^2}}{\operatorname{sh}\frac{x}{2}}dx+\int_\delta^\infty \frac{1-e^{-yx^2}}{\operatorname{sh}\frac{x}{2}}dx$$

for any $\delta > 1$, all three integrals being positive. Since $e^{-yx^2} \geq 1 - yx^2$,

$$\int_0^1 \frac{1-e^{-yx^2}}{\operatorname{sh}\frac{x}{2}} dx \leq \int_0^1 \frac{yx^2}{\operatorname{sh}\frac{x}{2}} dx = O(y).$$

By the same reason

$$\int_1^\delta \frac{1-e^{-yx^2}}{\operatorname{sh}\frac{x}{2}} dx \leq (\delta-1) \max_{1\leq x\leq\delta} \frac{1-e^{-yx^2}}{\operatorname{sh}\frac{x}{2}} \leq (\delta-1)\frac{1-e^{-y\delta^2}}{\operatorname{sh}\frac{1}{2}} \leq \frac{\delta-1}{\operatorname{sh}\frac{1}{2}} y\delta^2 \,.$$

For the third integral we have

$$\int_\delta^\infty \frac{1-e^{-yx^2}}{\operatorname{sh}\frac{x}{2}} dx \leq \int_\delta^\infty \frac{1}{\operatorname{sh}\frac{x}{2}} dx = O(e^{-\delta}) \,,$$

as δ tends to infinity. If δ tends to infinity in such a way that $y\delta^3$ tends to zero (e.g., put $\delta = y^{-1/4} = \log^{\frac{1}{4}} g$), these inequalities show that T_3 (and thus T_4) tends to zero.

For T_5 we have

$$0 \leq T_5 = \frac{2}{g}\int_0^\infty e^{-yx^2} \operatorname{ch}\frac{x}{2} dx \leq \frac{2}{g}\int_0^\infty e^{-yx^2+\frac{x}{2}} dx \leq \frac{2}{g}\int_{-\infty}^\infty e^{-yx^2+\frac{x}{2}} dx$$

$$= \frac{2}{g}\sqrt{\frac{\pi}{y}} e^{\frac{1}{16y}} = 2g^{-15/16}\sqrt{\pi \log g},$$

which shows that it also tends to zero.

Then it is sufficient to note that all terms in the sum T_6 are positive; indeed,

$$\int_{-\infty}^\infty e^{itx-yx^2} dx = e^{-t^2/4y}\int_{-\infty}^\infty e^{-y(x-\frac{it}{2y})^2} dx = \sqrt{\frac{\pi}{y}} e^{-t^2/4y} > 0 \,,$$

which finishes the proof.□

Recall that in the function field case we also have the Basic Inequality (cf. [12], [34], [36]), which is valid unconditionally:

Theorem 3.2.

$$\sum_{m=1}^\infty \frac{m\beta_m}{r^{m/2}-1} \leq 1$$

for any asymptotically exact family of function fields over $\mathbb{F}_r$. □

Question. *How large the difference* δ *between the right hand side and the left hand side of Theorems* 3.1 *and* 3.2 *can be?*

We call δ the *deficiency* of the family. Of course, $0 \leq \delta \leq 1$, and for asymptotically bad families $\delta = 1$. In the function field case, at least when r is an even prime power, families of modular curves provide examples with $\delta = 0$. In the number field case we do not know such a family. In fact, all known examples are those of unramified class field towers, sometimes with extra splitting conditions (see Section 9). Ihara [12] produced an example of a class field tower of the field $\mathbb{Q}(\sqrt{-3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot23\cdot31})$ with $\delta \leq 0.248...$

Then Yamamura [39] presented other examples of fields having infinite unramified class towers and very small δ's. The best of his examples would have $\delta \leq 0.088...$ The main tool of his paper is a theorem giving a condition for a field to have an infinite unramified class field tower. This theorem looks however not to be true, at least, as we are going to show in Section 9, it contradicts GRH. Unfortunately all the examples of [39] depend heavily on this theorem, and therefore cannot be considered as correct. In Section 9 we discuss the problem in more detail.

Let us remark that the class field tower of the Martinet field

$$\mathbb{Q}(\cos\frac{2\pi}{11}, \sqrt{2}, \sqrt{-23})$$

has $\delta \leq 0.1601...$, which was the best one known for many years. Quite recently Hajir and Maire [8] produced several better examples, the best one they get is given as follows. Let $K = \mathbb{Q}(\xi)$, where ξ is a root of the polynomial $x^8 - 9x^6 + 24182x^4 + 60281988x^2 + 895172213$, then K has an infinite class field tower ramified (tamely) only at two places over 5. This tower has $\delta \leq 0.141\ldots$.

3.2 Unconditional basic inequalities

Let us now give some partial results which can be obtained without assuming GRH. Unfortunately, they are much weaker.

Proposition 3.1 (Basic Inequality$'$). *For any asymptotically exact family of number fields one has*

$$2\sum_q \phi_q \log q \sum_{m=1}^{\infty} \frac{1}{q^m+1} + \phi_{\mathbb{R}}(\gamma/2 + 1/2 + \log 2\sqrt{\pi}) + \phi_{\mathbb{C}}(\gamma + \log 4\pi) \leq 1\ ,$$

the first sum being taken over all prime powers q.

Note that archimedean coefficients are

$$\alpha'_{\mathbb{R}} := \gamma/2 + 1/2 + \log 2\sqrt{\pi} \approx 2.054..., \quad \text{and} \quad \alpha'_{\mathbb{C}} := \gamma + \log 4\pi \approx 3.108...,$$

whereas in Theorem 3.1 they are

$$\alpha_{\mathbb{R}} = \gamma/2 + \pi/4 + \log 2\sqrt{2\pi} \approx 2.686..., \quad \text{and} \quad \alpha_{\mathbb{C}} = \gamma + \log 8\pi \approx 3.801...\ .$$

This result ameliorates the unconditional Odlyzko–Serre inequality.

Proof. The method of the proof is exactly the same as for the Basic Inequality (GRH Theorem 3.1) with the only difference in the choice of the function $F(x)$; without GRH we choose $F(x) = \frac{e^{-yx^2}}{\operatorname{ch}\frac{x}{2}}$ to have $\operatorname{Re}\Phi(s)$ positive on the whole critical strip, which is checked by an elementary calculation using the maximum principle (cf. [20], eq. 2.4). Using the Guinand–Weil explicit formula given at the beginning of Section 3 and dividing the obtained equality by $2g$ we get

$$1 = \frac{r_1}{g}(\frac{\pi}{4} + \frac{\gamma}{2} + \frac{\log 8\pi}{2}) + \frac{r_2}{g}(\gamma + \log 8\pi) - \frac{n}{g}\int_0^{\infty} \frac{1 - (e^{-yx^2}/\operatorname{ch}\frac{x}{2})}{4\operatorname{sh}\frac{x}{2}}dx$$

$$-\frac{r_1}{g}\int_0^\infty \frac{1-(e^{-yx^2}/\operatorname{ch}\frac{x}{2})}{4\operatorname{ch}\frac{x}{2}}dx - \frac{2}{g}\int_0^\infty e^{-yx^2}dx + \operatorname{Re}\frac{1}{2g}\sum_{u+it}\int_{-\infty}^\infty \frac{e^{itx+(u-\frac{1}{2})x-yx^2}}{\operatorname{ch}\frac{x}{2}}dx$$

$$+\sum_q \frac{N(q)}{g}\sum_{m=1}^\infty \frac{2q^{-m/2}e^{-ym^2\log^2 q}\log q}{q^{-m/2}+q^{m/2}}\ ,$$

where in the first sum $s=u+it$ runs over all zeroes of $\zeta_K(s)$ in the critical strip, the rest of the notation being that of the proof of GRH Theorem 3.1. Thus $1=T_1'+T_2'-T_3'(y)-T_4'(y)-T_5'(y)+T_6'(y)+T_7'(y)$ is presented as a sum of seven terms, some of which depend on the value of y. Note that $T_1'+T_2'=T_1+T_2$. We are going to show that if y tends to 0 exactly as described in the proof of GRH Theorem 3.1 then $T_3'(y)$ tends to $\frac{(\phi_{\mathbb{R}}+2\phi_{\mathbb{C}})\log 2}{2}$, $T_4(y)$ tends to $\phi_{\mathbb{R}}(\frac{\pi}{4}-\frac{1}{2})$, $T_5'(y)$ tends to 0, $T_6'(y)$ is non-negative, $T_7'(y)$ tending to $2\sum\limits_q \phi_q\log q\sum_{m=1}^\infty (q^m+1)^{-1}$, which proves the result. Indeed, the statement on T_6 is obvious from the very choice of $F(x)$, as explained above. The statement on $T_5'(y)$ follows from the bound $|T_5'(y)|\le T_5$ implied by the inequality $ch\frac{x}{2}\ge 1$. Note that here, as above, we take $y=\frac{1}{\log g}$. To prove the statements on $T_i'(y)$ for $i=3,4$ and 7 it is sufficient to notice that $T_3'(0)=\frac{n\log 2}{2g}$, $T_4'(0)=\frac{r_1}{g}(\frac{\pi}{4}-\frac{1}{2})$, which is an elementary calculation of integrals, $T_7'(0)=2\sum\limits_q\phi_q\log q\sum_{m=1}^\infty(q^m+1)^{-1}$, and that $|T_i'(y)-T_i'(0)|\le T_i$ for $i=3,4,7$, which follows from the same inequality $\operatorname{ch}\frac{x}{2}\ge 1$ as well. □

We shall also present the following weaker result, proved by Y.Ihara [12] for the case of unramified towers. It is sometimes easier to calculate with (cf. the proof of Theorem 9.7), and has a nice interpretation in terms of the limit zeta-function (cf. Remark 4.3).

Proposition 3.2 (Basic Inequality″). *For any asymptotically exact family of number fields one has*

$$\sum_q \frac{\phi_q\log q}{q-1}+(\gamma/2+\log 2\sqrt{\pi})\phi_{\mathbb{R}}+(\gamma+\log 2\pi)\phi_{\mathbb{C}}\le 1\ .$$

Note that archimedean coefficients are

$$\alpha_{\mathbb{R}}'':=\gamma/2+\log 2\sqrt{\pi}\approx 1.554...,\quad \text{and}\quad \alpha_{\mathbb{C}}'':=\gamma+\log 2\pi\approx 2.415...\ .$$

Proof. To prove the result one uses Stark's formula (cf. [20], p. 120)

$$\log|D|=r_1(\log\pi-\psi(s/2))+2r_2(\log(2\pi)-\psi(s))-\frac{2}{s}-\frac{2}{s-1}$$

$$+2{\sum_\rho}'\frac{1}{s-\rho}+2\sum_P\sum_{m=1}^\infty N(P)^{-ms}\log N(P)\ ,$$

where in the first sum ρ runs over the zeroes of $\zeta_K(s)$ in the critical strip, and $\sum'$ means that the ρ and $\bar\rho$ are to be grouped together, while the external sum in the double sum is taken over all prime divisors P of K, and $\psi(s)=\Gamma'(s)/\Gamma(s)$. We then apply this formula to $K=K_j$ for $s=1+\frac{1}{\sqrt{g_j}}$, where as usual $g_j=\log\sqrt{|D_j|}$, and divide it by $2g_j$. We get

$$1=\frac{r_1}{2g_j}(\log\pi-\psi(s/2))+\frac{r_2}{g_j}(\log 2\pi-\psi(s))-\frac{1}{\sqrt{g_j}(1+\sqrt{g_j})}-\frac{1}{\sqrt{g_j}}$$
$$+\frac{1}{g_j}\sum_{\rho}{}'\frac{1}{s-\rho}+\frac{1}{g_j}\sum_P\sum_{m=1}^{\infty}N(P)^{-ms}\log N(P).$$

When g_j grows, the first two terms tend, respectively, to $\alpha''_{\mathbb{R}}\phi_{\mathbb{R}}$ and $\alpha''_{\mathbb{C}}\phi_{\mathbb{C}}$ since $\psi(\frac{1}{2})=-\gamma-\log 4$ and $\psi(1)=-\gamma$; the third and the forth terms tend to zero, the fifth being positive, and the last term tends to $\sum\limits_q\frac{\phi_q\log q}{q-1}$, which finishes the proof. □

Proposition 3.2 ameliorates Stark's inequality

$$\alpha''_{\mathbf{R}}\phi_{\mathbb{R}}+\alpha''_{\mathbb{C}}\phi_{\mathbb{C}}\le 1.$$

4 Zeta function

We define the *limit zeta function* of an asymptotically exact family $\mathcal{K}$ as

$$\zeta_{\mathcal{K}}(s)=\zeta_\phi(s):=\prod_q(1-q^{-s})^{-\phi_q},$$

and its *completed limit zeta function* as

$$\tilde\zeta_{\mathcal{K}}(s)=\tilde\zeta_\phi(s):=e^s2^{-\phi_{\mathbb{R}}}\pi^{-s\phi_{\mathbb{R}}/2}(2\pi)^{-s\phi_{\mathbb{C}}}\Gamma(\frac{s}{2})^{\phi_{\mathbb{R}}}\Gamma(s)^{\phi_{\mathbb{C}}}\prod_q(1-q^{-s})^{-\phi_q}$$

in the number field case, and

$$\tilde\zeta_{\mathcal{K}}(s)=\tilde\zeta_\phi(s):=r^s\prod_q(1-q^{-s})^{-\phi_q}$$

in the function field case; q runs over all prime powers in the number field case and over powers of r in the function field case. As usual, by raising to a complex power a function in s defined for $\operatorname{Re}s>a\ge 0$ and such that its values are real positive for real $s>a$, we mean unique analytic continuation of what is real positive for real s. For an expression $(1-q^{-s})^{-\phi_q}$ this is the same as to take the value given by the binomial series.

Note that our definition of $\tilde\zeta_\phi$ slightly differs in the function field case from that of [36]. Note also that $\zeta_\phi(s)$ and $\tilde\zeta_\phi(s)$ depend only on $\phi=\{\phi_\alpha\}$ and do not depend on the particular sequence of global fields.

GRH Proposition 4.1. *For any asymptotically exact family of global fields the product defining ζ_ϕ (and $\tilde\zeta_\phi$) converges absolutely in the closed half-plane*

$\mathrm{Re}(s) \geq \frac{1}{2}$*, and defines an analytic function in* $\mathrm{Re}(s) > \frac{1}{2}$*. In the function field case the result is unconditional.*

Proof. The product converges absolutely if and only if the series

$$\sum_q \phi_q \log |\frac{1}{1-q^{-s}}|$$

does, but for $\mathrm{Re}(s) \geq 1/2$ one has

$$\sum_q \phi_q \log |\frac{1}{1-q^{-s}}| \leq \sum_q \phi_q \log \frac{1}{1-q^{-1/2}} \ ,$$

which in its turn converges, since, starting from some q,

$$\phi_q \log \frac{1}{1-q^{-1/2}} \leq \frac{\phi_q \log q}{\sqrt{q}-1},$$

and the series

$$\sum_q \frac{\phi_q \log q}{\sqrt{q}-1}$$

converges by GRH Theorem 3.1. $\square$

Remark 4.1. In fact since the coefficients of the Dirichlet series corresponding to ζ_ϕ are positive and $\zeta_\phi(1/2)$ is finite, the product converges absolutely for $\mathrm{Re}(s) > \frac{1}{2} - \varepsilon(\phi)$, where $\varepsilon(\phi)$ depends on how large ϕ_q are. It can even happen that $\tilde{\zeta}_\phi(s)$ is an entire function. For asymptotically bad families we have $\tilde{\zeta}_\phi(s) = e^s$ or r^s depending on the case under consideration. In fact, we do not know a single example of an infinite global field for which the product does not converge in the half-plane $\mathrm{Re}(s) > 0$; note also that the archimedean factors are analytic in this half-plane.

Let now

$$\xi_\phi(s) := (\log \tilde{\zeta}_\phi)' = \tilde{\zeta}'_\phi / \tilde{\zeta}_\phi =$$
$$1 - \frac{\phi_{\mathbb{R}}}{2} \log \pi - \phi_{\mathbb{C}} \log 2\pi + \frac{1}{2}\phi_{\mathbb{R}}\psi(\frac{s}{2}) + \phi_{\mathbb{C}}\psi(s) - \sum_q \phi_q \frac{\log q}{q^s - 1}$$

in the number field case (where, as before, $\psi(s) = \frac{\Gamma'}{\Gamma}(s)$), and

$$\xi_\phi(s) := (\log_r \tilde{\zeta}_\phi)' = \frac{1}{\log r}\left(\frac{\tilde{\zeta}'_\phi}{\tilde{\zeta}_\phi}\right) = 1 - \sum_{m=1}^{\infty} \frac{m\phi_{r^m}}{r^{ms}-1}$$

in the function field case.

Then one can express the Basic Inequality (GRH Theorem 3.1 and Theorem 3.2) as

$$\xi_\phi(1/2) \geq 0 \ ,$$

and the Generalized Brauer–Siegel Theorem (GRH Theorem 7.2 and Theorem 7.3 below) either as

$$\lim_{i\to\infty} \frac{\log h_i R_i}{g_i} = \log \tilde{\zeta}_\phi(1),$$

or as

$$\lim_{i\to\infty}\frac{\log\varkappa_i}{g_i}=\log\zeta_\phi(1),$$

where $\varkappa_i$ is the residue of $\zeta_{K_i}(s)$ at 1. The function field case of the Generalized Brauer–Siegel Theorem reads either as

$$\lim_{i\to\infty}\frac{\log_r h_i}{g_i}=\log_r(\tilde{\zeta}_\phi(1))$$

or as

$$\lim_{i\to\infty}\frac{\log_r\varkappa_i}{g_i}=\log_r(\zeta_\phi(1))\ .$$

Unconditionally, we have

Proposition 4.2. *a) For any asymptotically exact family of global fields the product defining ζ_ϕ (and $\tilde{\zeta}_\phi$) converges absolutely in the closed half-plane* $\operatorname{Re} s\geq 1$ *, and defines an analytic function in* $\operatorname{Re} s>1$ *.*

b) For $\operatorname{Re} s>1$ *we have the pointwise limits*

$$\zeta^0_\phi(s)=\lim_{j\to\infty}\zeta_{K_j}(s)^{1/g_j},$$

$$\tilde{\zeta}_\phi(s)=\lim_{j\to\infty}\tilde{\zeta}_{K_j}(s)^{1/g_j},$$

where $\tilde{\zeta}_{K_j}(s)$ is the completed zeta-function defined by

$$\tilde{\zeta}_{K_j}(s)=\mid D_j\mid^{s/2}2^{-r_1(K_j)}\pi^{-sr_1(K_j)/2}(2\pi)^{-sr_2(K_j)}\Gamma(s/2)^{r_1(K_j)}\Gamma(s)^{r_2(K_j)}\zeta_{K_j}(s)$$

in the number field case, and by

$$\tilde{\zeta}_{K_j}(s)=r^{s(g-1)}\zeta_{K_j}(s)$$

in the function field case. For any $\varepsilon>0$ the convergence is uniform in the half-plane $\operatorname{Re} s>1+\varepsilon$ *(and, thus on compact subsets in the half-plane* $\operatorname{Re} s>1$*).*

c) Let $s_0>1$, $s_0\neq 2,4$, be a real number such that the limit

$$\lim_{j\to\infty}\tilde{\zeta}_{K_j}(s_0)^{1/g_j}$$

exists for some family $\{K_j\}$. Then the family is asymptotically exact.

Proof. The proof of a) is the same as that of GRH Proposition 4.1, but in place of GRH Theorem 3.1 we use Proposition 3.2.

Let us prove b). The proof is essentially the same as in the function fields case considered in [36]. Let K_j be a field from our family. Note that it is sufficient to consider the case of real s, since $\mid\zeta_{K_j}(x)^{1/g}/\zeta_\phi(x)-1\mid\leq\mid\zeta_{K_j}(s)^{1/g}/\zeta_\phi(s)-1\mid$ if $x=s+it$ with $s>1$ (look at the corresponding Dirichlet series). For a real $s>1$ we have

$$\zeta_{K_j}(s)^{1/g}/\zeta_\phi(s)=\prod_q(1-q^{-s})^{N_q(K_j)/g_j-\phi_q}.$$

Let g_0 be a positive integer such that for any $q \le M$ (here M is a positive integer to be specified below) one has $|\phi_q - N_q(K_j)/g_j| \le \delta_1$ for $g_j \ge g_0$, where a real positive δ_1 is also to be specified. Then we have

$$\prod_{q\le M}(1-q^{-s})^{\delta_1} \le \prod_{q\le M}(1-q^{-s})^{N_q(K_j)/g_j-\phi_q} \le \prod_{q\le M}(1-q^{-s})^{-\delta_1}.$$

For any $s \ge 1+\epsilon$ the product $\prod_{q\le M}(1-q^{-s})$ satisfies the inequalities

$$F_M(\epsilon)^{-1} \le \prod_{q\le M}(1-q^{-s}) \le F_M(\epsilon)$$

for some real $F_M(\epsilon) > 1$. Thus

$$F_M(\epsilon)^{-\delta_1} \le \prod_{q\le M}(1-q^{-s})^{N_q(K_j)/g_j-\phi_q} \le F_M(\epsilon)^{\delta_1}.$$

Let us now estimate the "tail" $\prod_{q\ge M+1}(1-q^{-s})^{N_q(K_j)/g_j-\phi_q}$.

We show first that always $|N_q(K_j)/g_j - \phi_q| \le 3q$ for sufficiently large q. Indeed, in the function field case we have

$$N_q(K_j)/g_j \le \frac{q+1+2g_jq^{1/2}}{g_j} \le 2q$$

and $\phi_q \le (q^{1/2}-1)/m < q^{1/2}$ by the Basic Inequality, which proves the assertion in this case. In the number field case $N_q(K_j) \le n_j = \deg K_j \le \text{const}\cdot g_j$ and thus $N_q(K_j)/g_j$ is bounded by a constant; on the other hand the unconditional Basic Inequality implies that $\phi_q < q$, which proves the assertion (with much room to spare) in the number field case.

The assertion implies that the tail lies between

$$G(\epsilon, M) = \prod_{q\ge M+1}^{\infty}\left(1-q^{-(1+\epsilon)}\right)^{3q}$$

and its inverse.

Since $(1-q^{-(1+\epsilon)})^{3q} = \left((1-q^{-(1+\epsilon)})^{3q^{(1+\epsilon)}}\right)^{q^{-\epsilon}}$ and $(1-q^{-(1+\epsilon)})^{3q^{(1+\epsilon)}}$ tends to e^{-3} for $q\to\infty$ we get the tail to be between $C\exp\Big(\sum\limits_{q\ge M+1}^{\infty} q^{-\epsilon}\Big)$ and its inverse for any $C>1,\ s\ge 1+\epsilon$, and M sufficiently large. Choosing C and δ_1 such that $F_M(\epsilon)^{\delta_1} \le \sqrt{1+\delta}$ and M such that $C\exp\Big(\sum\limits_{q\ge M+1}^{\infty} q^{-\epsilon}\Big) \le \sqrt{1+\delta}$, we get the result.

Let us prove c). Let us suppose that the family is not asymptotically exact; it means that there exists α such that the sequence $N_\alpha(K_j)/g_j$ has at least two different limit points, which we denote ϕ'_α and ϕ''_α. We can choose two asymptotically exact subfamilies $\mathcal{K}'$ and $\mathcal{K}''$ of our family such that $\phi_\beta(\mathcal{K}') =$

$\phi_\beta(\mathcal{K}'')$ for all $\beta \neq \alpha$ and $\phi_\alpha(\mathcal{K}') = \phi'_\alpha$, $\phi_\alpha(\mathcal{K}'') = \phi''_\alpha$. Then using b) we see that our condition implies $(f_\alpha(s_0))^{\phi'_\alpha} = (f_\alpha(s_0))^{\phi''_\alpha}$ where f_α is the factor in the product defining $\tilde{\zeta}_\phi$, corresponding to α. Since $f_\alpha(s_0) \neq 1$ for $s_0 \neq 2, 4$ (for non-archemedean factors this is true for any s_0, but for the gamma factors one needs to throw out the values $s_0 = 2, 4$) we get the result. $\square$

Remark 4.2. Let us sketch briefly another way to prove Proposition 4.2, b. Each $\zeta_{K_j}(s)^{1/g_j}$ can be expressed as an ordinary Dirichlet series $\sum_{m=1}^{\infty} a_j(m)m^{-s}$ convergent for $\mathrm{Re}(s) > 1$ if the terms in the Euler product, raised to the $1/g_j$ power are expanded using the binomial theorem. For each fixed $m \geq 1$ the sequence $a_j(m)$, $j = 1, 2, \ldots$, goes to a limit as $j \to \infty$.

Remark 4.3. Proposition 3.2 can be rewritten as $\xi_\phi(1) \geq 0$.

Remark 4.4. Part b) of Proposition 4.2 shows in which sense the sequence of zeta-functions of an asymptotically exact family of global fields tends to the limit zeta-function.

Remark 4.5. The definition of $\tilde{\zeta}_\phi(s)$ in the number field case is chosen on the one hand so as to write the Basic Inequality and the Generalized Brauer–Siegel Theorem in the shortest possible way, and on the other hand so that it is the natural analogue of the function

$$\tilde{\zeta}_K(s) = |D|^{s/2} 2^{-r_1} \pi^{-sr_1/2} (2\pi)^{-sr_2} \Gamma(s/2)^{r_1} \Gamma(s)^{r_2} \zeta_K(s)$$

which is invariant under $s \mapsto 1 - s$. Note however, that the condition to be invariant under $s \mapsto 1 - s$ does not change if the function is multiplied by a constant, and our function $\tilde{\zeta}_K(s)$ differs from the function $\Lambda_K(s)$ used in [16] by the factor 2^{-r_1}. The above formulae strongly suggest our normalization to be the natural one.

Remark 4.6. Comparing the definitions and results for the number field case and for the function field one, we conclude that the "ground field" of a number field "is" of cardinality $e \approx 2.718281828459045...$

5 Zeta zeroes and the explicit formula

We are going to study the asymptotic distribution of zeroes of ζ_K for g_K tending to infinity. We start with number fields and suppose GRH to hold. Note that in the case of asymptotically bad families this result was essentially obtained in [15].

5.1 Number field case

Let us recall several standard notions and facts from the theory of distributions (cf. [28]). Let $\mathcal{S} = \mathcal{S}(\mathbb{R})$ be the space of complex valued infinitely differentiable functions on $\mathbb{R}$ which are rapidly (i.e., faster than any polynomial) decreasing together with all their derivatives. This vector space is naturally equipped with a standard topology, so that the Fourier transform is a topological automorphism of $\mathcal{S}$. Its topological dual $\mathcal{S}'$ is called the space of tempered distributions. By duality, the Fourier transform is also defined on $\mathcal{S}'$ and it is also a topological automorphism there. The space $\mathcal{S}'$ is contained in the space $\mathcal{D}'$ of all distributions, which is the topological dual of the space $\mathcal{D}$ of complex valued infinitely differentiable functions with compact support on $\mathbb{R}$. The space of measures $\mathcal{M}$ is the topological dual of the space of complex valued continuous functions with compact support on $\mathbb{R}$. Of course, $\mathcal{M} \subset \mathcal{D}'$. The space of measures $\mathcal{M}$ contains the cone of positive measures $\mathcal{M}_+$, i.e., of those measures whose value at a positive real-valued function is positive. The space of distributions $\mathcal{D}'$ also contains the cone of positive distributions $\mathcal{D}'_+$. It is known that $\mathcal{D}'_+ = \mathcal{M}_+$ (cf. [28], Thm.V of Ch.I). The intersection $\mathcal{M}_{sl} = \mathcal{M} \cap \mathcal{S}'$ is called the space of measures of slow growth. The criterion for a measure to be of slow growth is that for some positive integer k the integral

$$I_k = \int_{-\infty}^{\infty} (x^2+1)^{-k} d\mu$$

converges (cf. [28], Thm.VII of Ch.VII).

Let $\mathcal{F} = \{K_j\}$ be an asymptotically exact family of number fields. For each K_j we define the measure

$$\Delta_{K_j} := \frac{\pi}{g_j} \sum_{\zeta_{K_j}(\rho)=0} \delta_{t(\rho)} \ ,$$

where $g_j := g_{K_j}$, $t(\rho) = (\rho - \frac{1}{2})/i$, and ρ runs over all non-trivial zeroes of the zeta-function $\zeta_{K_j}(s)$; here δ_a denotes the atomic (Dirac) measure at a. Because of GRH $t(\rho)$ is real, and Δ_{K_j} is a discrete measure on $\mathbb{R}$. Moreover, Δ_{K_j} is a measure of slow growth, which follows, e.g., from the Weil Explicit Formula (see the proof of GRH Theorem 5.1 below).

Now we are ready to formulate the main result of this section, expressing the limit distribution of zeta zeroes in terms of the parameters $\phi = \{\phi_\alpha\}$ of the asymtotically exact family.

GRH Theorem 5.1 (GRH Explicit Formula). *For an asymptotically exact family $\mathcal{K}$, in the space of measures of slow growth on $\mathbb{R}$ the limit*

$$\Delta = \Delta_{\mathcal{K}} := \lim_{j\to\infty} \Delta_{K_j}$$

exists. Moreover, the measure Δ has a continuous density M_ϕ ,

$$M_\phi(t) = \mathrm{Re}\left(\xi_\phi\left(\frac{1}{2}+it\right)\right) =$$

$$1 - \sum_q \phi_q h_q(t) \log q + \frac{1}{2}\phi_{\mathbb{R}} \,\mathrm{Re}\,\psi\left(\frac{1}{4}+\frac{it}{2}\right) + \phi_{\mathbb{C}} \,\mathrm{Re}\,\psi\left(\frac{1}{2}+it\right) - \frac{\phi_{\mathbb{R}}}{2}\log\pi - \phi_{\mathbb{C}}\log 2\pi$$

where

$$h_q(t) = \frac{\sqrt{q}\cos(t\log q) - 1}{q+1-2\sqrt{q}\cos(t\log q)}, \quad \psi(s) = \frac{\Gamma'}{\Gamma}(s) \ .$$

Remark 5.6. This density depends only on the parameters $\phi = \{\phi_\alpha\}$.
GRH Corollary 5.1 (GRH Basic Equality). *We have*

$$\sum_q \frac{\phi_q \log q}{\sqrt{q}-1} + \phi_{\mathbb{R}}(\log\sqrt{8\pi} + \frac{\pi}{4} + \frac{\gamma}{2}) + \phi_{\mathbb{C}}(\log 8\pi + \gamma) = 1 - M_\phi(0) \ ,$$

in other words, the deficiency

$$\delta_{\mathcal{K}} = \xi_\phi(\frac{1}{2}) = M_\phi(0) \ .$$

Proof of GRH Corollary 5.1. Put $t = 0$ in the formula for $M_\phi(t)$.□

Proof of GRH Theorem 5.1. Here is the strategy of the proof: First we are going to prove that Δ exists as a limit in $\mathcal{S}'$, therefore, being positive, it lies in $\mathcal{M}_{sl,+} = \mathcal{S}'_+$. The next point is to show that this measure is absolutely continuous, i.e., of the form $F(t)dt$; to do it we have to prove that neither skyscraper, nor singular component occurs. Then we shall compare this measure $\Delta = F(t)dt$ with the measure $\Delta_0 = M_\phi(t)dt$ of the theorem: we first show that they coincide on the set of some specific functions $H_{y,a}(t)$, and then that this is enough for the measures to be equal.

We begin by proving that Δ is well-defined as a tempered distribution. We are going to use the Weil Explicit Formula in the form presented in [24], Section 1. We use the notation of Section 3; here we suppose that $F \in \mathcal{S}(\mathbb{R})$, and that it satisfies the condition

$$F(x), F'(x) \leq ce^{-(1/2+\varepsilon)|x|} \text{ as } |x| \to \infty. \qquad (*)$$

Note that for $s = \frac{1}{2} + it$

$$\Phi(s) := \int_{-\infty}^{\infty} F(x)e^{(s-1/2)x}dx = \hat{F}(t) \ ,$$

where

$$\hat{F}(t) = \int_{-\infty}^{\infty} F(x)e^{itx}dx \in \mathcal{S}(\mathbb{R})$$

is the Fourier transform of F.

The abovementioned Weil Explicit Formula reads as follows:

Let K be a number field, then the limit

$$\sum{}' \Phi(\rho) := \lim_{T\to\infty} \sum_{|\rho|<T} \Phi(\rho) = \lim_{T\to\infty} \sum_{\substack{|\frac{1}{2}+it|<T \\ \zeta_K(\frac{1}{2}+it)=0}} \hat{F}(t)$$

exists, where in the sum ρ runs over the set of zeroes of $\zeta_K(s)$ on the critical line $\mathrm{Re}(s) = 1/2$ *(which is supposed to be the set of all critical zeroes of $\zeta_K(s)$), and we have the following formula*:

$$\sum{}' \Phi(\rho) - \Phi(0) - \Phi(1) = F(0)(\log|D_K| - r_1 \log \pi - 2r_2 \log 2\pi)$$

$$- \sum_P \sum_{m=1}^{\infty} N(P)^{-m/2}[F(m \log N(P)) + F(-m \log N(P))] \log N(P) +$$

$$\frac{r_1}{2\pi} \int_{-\infty}^{\infty} \frac{\hat{F}(t) + \hat{F}(-t)}{2} \operatorname{Re} \psi\left(\frac{1}{4} + \frac{it}{2}\right) dt + \frac{r_2}{\pi} \int_{-\infty}^{\infty} \frac{\hat{F}(t) + \hat{F}(-t)}{2} \operatorname{Re} \psi\left(\frac{1}{2} + it\right) dt\ ,$$

where the external sum is taken over all primes P of K, and $N(P)$ denotes the absolute norm of P.

This is exactly the formula of [24], Section 1. We apply the formula only to functions from $\mathcal{S}(\mathbb{R})$ which clearly satisfy the other conditions of the theorem of [24], Section 1. Though there the function F is assumed to be even, we can apply the result to any function F from $\mathcal{S}(\mathbb{R})$, replacing it by $\frac{F(t)+F(-t)}{2}$.

One can rewrite the sum

$$\sum_P \sum_{m=1}^{\infty} N(P)^{-m/2}[F(m \log N(P)) + F(-m \log N(P))] \log N(P)$$

as

$$\sum_q N_q(K) \sum_{m=1}^{\infty} q^{-m/2}[F(m \log q) + F(-m \log q)] \log q,$$

the sum being taken over all prime powers q.

Let $\hat{\mathcal{D}} \subset \mathcal{S}$ be the Fourier dual of $\mathcal{D}$; it is a dense subspace of $\mathcal{S}$ (since $\mathcal{D}$ is dense in $\mathcal{S}$). Let $K = K_j$ be a field from our family. Take any $f \in \hat{\mathcal{D}}$ and let $f = \hat{F}$, $F \in \mathcal{D}$. We have $f(t) = \Phi(\frac{1}{2} + it)$. The function F satisfies the above condition (*), and, dividing by $2g_{K_j} = \log|D_{K_j}|$, we get:

$$\Delta_{K_j}(f) = \frac{\pi}{g_{K_j}} \sum_{\zeta_K(\frac{1}{2}+it)=0}{}' f(t) =$$

$$2\pi\left(\frac{\Phi(0) + \Phi(1)}{2g_{K_j}} + F(0)\left(1 - \frac{r_1}{2g_{K_j}} \log \pi - \frac{r_2}{g_{K_j}} \log 2\pi\right)\right)$$

$$-2\pi\left(\sum_q \frac{N_q(K_j)}{2g_{K_j}} \sum_{m=1}^{\infty} q^{-m/2}[F(m \log q) + F(-m \log q)] \log q\right) +$$

$$\frac{r_1}{2g_{K_j}} \int_{-\infty}^{\infty} \frac{f(t) + f(-t)}{2} \operatorname{Re} \psi\left(\frac{1}{4} + \frac{it}{2}\right) dt + \frac{r_2}{g_{K_j}} \int_{-\infty}^{\infty} \frac{f(t) + f(-t)}{2} \operatorname{Re} \psi\left(\frac{1}{2} + it\right) dt\ .$$

If we fix f and tend j to infinity then the right hand side tends to

$$\Delta(f) := 2\pi F(0)\left(1 - \frac{\phi_{\mathbb{R}}}{2} \log \pi - \phi_{\mathbb{C}} \log 2\pi\right)$$

$$-\pi \sum_q \phi_q \sum_{m=1}^{\infty} q^{-m/2}[F(m\log q)+F(-m\log q)]\log q+$$

$$\frac{\phi_{\mathbb{R}}}{2}\int_{-\infty}^{\infty}\frac{f(t)+f(-t)}{2}\,\mathrm{Re}\,\psi\left(\frac{1}{4}+\frac{it}{2}\right)dt+\phi_{\mathbb{C}}\int_{-\infty}^{\infty}\frac{f(t)+f(-t)}{2}\,\mathrm{Re}\,\psi\left(\frac{1}{2}+it\right)dt\ ,$$

since the family is asymptotically exact. This shows the existence of the limit

$$\Delta(f)=\lim_{j\to\infty}\Delta_{K_j}(f)$$

for any $f\in\hat{\mathcal{D}}$. The map $f\mapsto\Delta(f)$ is obviously linear. Thus in order to prove that Δ is a tempered distribution it is sufficient to verify that Δ is continuous on $\hat{\mathcal{D}}$ in the topology of $\mathcal{S}$, since $\hat{\mathcal{D}}$ is dense in $\mathcal{S}$. Let now $\{f_i\}$, $i=1,2,...$, be a sequence with $f_i\in\hat{\mathcal{D}}$ tending to zero in the topology of $\mathcal{S}$. Since the Fourier transform is a topological automorphism of $\mathcal{S}$, we conlude that the sequence $\{F_i\}$, where $f_i=\hat{F}_i$, tends to zero as well. In particular, both sequences $\{f_i\}$ and $\{F_i\}$ tend to zero uniformly. Let us then show that Δ is continuous. Indeed,

$$\Delta(f_i)=T_1(f_i)-T_2(f_i)+T_3(f_i)+T_4(f_i)\ ,$$

where

$$T_1(f_i)=2\pi F_i(0)\left(1-\frac{\phi_{\mathbb{R}}}{2}\log\pi-\phi_{\mathbb{C}}\log 2\pi\right)=\delta(\hat{f}_i)\left(1-\frac{\phi_{\mathbb{R}}}{2}\log\pi-\phi_{\mathbb{C}}\log 2\pi\right)\ ,$$

$$T_2(f_i)=\pi\sum_q \phi_q\sum_{m=1}^{\infty}q^{-m/2}[F_i(m\log q)+F_i(-m\log q)]\log q\ ,$$

$$T_3(f_i)=\frac{\phi_{\mathbb{R}}}{2}\int_{-\infty}^{\infty}\frac{f_i(t)+f_i(-t)}{2}\,\mathrm{Re}\,\psi\left(\frac{1}{4}+\frac{it}{2}\right)dt\ ,$$

and

$$T_4(f_i)=\phi_{\mathbb{C}}\int_{-\infty}^{\infty}\frac{f_i(t)+f_i(-t)}{2}\,\mathrm{Re}\,\psi\left(\frac{1}{2}+it\right)dt\ .$$

The term $T_1(f_i)$ is clearly continuous as well as the terms $T_3(f_i)$ and $T_4(f_i)$, since the measures $\mathrm{Re}\,\psi\left(\frac{1}{2}+it\right)dt$ and $\mathrm{Re}\,\psi\left(\frac{1}{4}+\frac{it}{2}\right)dt$ are of slow growth (cf. [16], XVIII.1). To prove the continuity of $T_2(f_i)$ one notes that from the GRH version of the Basic Inequality (GRH Theorem 3.1 above) it follows that

$$T_2(f_i)\leq 2\pi\sup_{x\in\mathbb{R}}|F_i(x)|$$

which finishes the proof of the continuity of Δ.

Thus we see that Δ is a tempered distribution, $\Delta\in\mathcal{S}'$. Note that by its very definition this distribution is positive, i.e., $\Delta(f)$ is real non-negative for a real non-negative $f\in\mathcal{S}(\mathbb{R})$. Since any positive distribution is a (positive) measure, one concludes that Δ is a measure of slow growth. Then we have to show that $\Delta=M_\phi dx$. To do that we need the following lemmata.

For any $a \in \mathbb{R}$ and any $y > 0$ let us define the function $H_{y,a}(x) \in \mathcal{S}(\mathbb{R})$

$$H_{y,a}(x) := \frac{1}{2\sqrt{\pi y}} \exp(\frac{-(a-x)^2}{4y})$$

so that

$$F_{y,a}(x) = \frac{\hat{H}_{y,a}(x)}{2\pi} := \frac{1}{2\pi} \exp(-yx^2 + iax).$$

Lemma 5.1. *We have*

$$\lim_{y \to +0} \Delta(H_{y,a}) = M_\phi(a) \ .$$

Note that for y tending to zero the function $H_{y,a}$ tends to δ_a in the sense of distributions, where δ_a is the Dirac measure concentrated in a.

Lemma 5.2. *Let μ be a positive mesure on $\mathbb{R}$ such that for any $a \in \mathbb{R}$ one has*

$$\lim_{y \to +0} \mu(H_{y,a}) = M(a) \tag{2}$$

with a function M continuous on $\mathbb{R}$. Then $\mu = M(x)dx$.

One notes that the theorem follows from these two lemmata. Let us now prove them.

Proof of Lemma 5.1. Let us apply the explicit formula to $f = H_{y,a}$. Since $H_{y,a}$ is even, we get

$$\Delta_{K_j}(H_{y,a}) = \frac{\pi(H_{y,a}(0) + H_{y,a}(1))}{g_{K_j}} + 2\pi F_{y,a}(0) \left(1 - \frac{r_1}{2g_{K_j}} \log \pi - \frac{r_2}{g_{K_j}} \log 2\pi\right) -$$

$$\sum_q \frac{N_q(K_j)}{2g_{K_j}} \sum_{m=1}^{\infty} q^{-m/2} [\hat{H}_{y,a}(m \log q) + \hat{H}_{y,a}(-m \log q)] \log q +$$

$$\frac{r_1}{2g_{K_j}} \int_{-\infty}^{\infty} H_{y,a}(t) \operatorname{Re} \psi\left(\frac{1}{4} + \frac{it}{2}\right) dt + \frac{r_2}{g_{K_j}} \int_{-\infty}^{\infty} H_{y,a}(t) \operatorname{Re} \psi\left(\frac{1}{2} + it\right) dt \ .$$

Since

$$\Delta(H_{y,a}) = \lim_{j \to \infty} \Delta_{K_j}(H_{y,a}),$$

$$2\pi F_{y,a}(0) = 1,$$

we get

$$\Delta(H_{y,a}) = 1 - \frac{\phi_{\mathbb{R}}}{2} \log \pi - \phi_{\mathbb{C}} \log 2\pi - \frac{1}{2} \sum_q \phi_q \log q \sum_{m=1}^{\infty} q^{-m/2} e^{-y(m \log q)^2} (q^{iam} + q^{-iam}) +$$

$$\frac{\phi_{\mathbb{R}}}{2} \int_{-\infty}^{\infty} H_{y,a}(t) \operatorname{Re} \psi\left(\frac{1}{4} + \frac{it}{2}\right) dt + \phi_{\mathbb{C}} \int_{-\infty}^{\infty} H_{y,a}(t) \operatorname{Re} \psi\left(\frac{1}{2} + it\right) dt \ .$$

Let us tend y to zero. Then the expression $e^{-y(m\log q)^2}(q^{iam}+q^{-iam})$ tends to $q^{iam}+q^{-iam}=2\cos(am\log q)$ and thus

$$\frac{1}{2}\sum_q \phi_q \log q \sum_{m=1}^{\infty} q^{-m/2} e^{-y(m\log q)^2}(q^{iam}+q^{-iam}) \to \sum_q \phi_q h_q(a)\log q\ ,$$

since $h_q(a)=\sum_{m=1}^{\infty} q^{-m/2}\cos(am\log q)$. Since $H_{y,a}(t)$ tends to δ_a in the space $\mathcal{M}$, and $\operatorname{Re}\psi\left(\frac{1}{4}+\frac{it}{2}\right)$ is a C^∞-function, $H_{y,a}(t)\operatorname{Re}\psi\left(\frac{1}{4}+\frac{it}{2}\right)$ tends to $\delta_a \operatorname{Re}\psi(\frac{1}{4}+i\frac{a}{2})$, and thus

$$\frac{\phi_{\mathbb{R}}}{2}\int_{-\infty}^{\infty} H_{y,a}(t)\operatorname{Re}\psi\left(\frac{1}{4}+\frac{it}{2}\right)dt \to \frac{\phi_{\mathbb{R}}}{2}\operatorname{Re}\psi\left(\frac{1}{4}+\frac{ia}{2}\right)$$

for y tending to zero. A similar argument shows that the term

$$\phi_{\mathbb{C}}\int_{-\infty}^{\infty} H_{y,a}(t)\operatorname{Re}\psi\left(\frac{1}{2}+it\right)dt \to \phi_{\mathbb{C}}\operatorname{Re}\psi\left(\frac{1}{2}+ia\right)$$

for $y\to 0$, which finishes the proof.□

Proof of Lemma 5.2. Since μ is a positive measure one can write $\mu = dG$ for a non-decreasing function G, and the standard decomposition $G=G_0+G_1+G_2$ with absolutely continuous G_0, singular G_1 and a jump-function G_2 (cf. [13] Ch. VI, section 4) shows that $\mu = dG = dG_0 + dG_1 + dG_2$. Let us prove that the property (1) implies $G_2=G_1=0$. Indeed, it is sufficient to show that

$$\int H_{y,a}(t)dG_i, \quad i=1,2$$

cannot be bounded for $y\to +0$. For G_2 it is almost obvious, since

$$dG_2=\sum_n s_n\delta_{t_n},$$

where

$$G_2(t)=\sum_{n:t_n\le t} s_n,$$

and δ_{t_n} is the Dirac measure at t_n. If $G_2\ne 0$, i.e., if the sum is non-empty, let $s_i>0$ and consider

$$\int H_{y,t_i}(t)dG_2=\sum_n s_n H_{y,t_i}(t_n)\ge s_i H_{y,t_i}(t_i)=\frac{s_i}{2\sqrt{\pi y}},$$

which obviously tends to infinity for $y\to +0$. Thus, $G_2=0$. If $G_1\ne 0$ then there exist $a,b\in\mathbb{R}$, $a<b$, with $G_1(a)<G_1(b)$. Recall that since G_1 is singular, its derivative G_1' is zero almost everywhere. Let us now show that there exists $x_0\in[a,b]$ such that

$$\limsup_{\varepsilon\to+0}\frac{G_1(x_0+\varepsilon)-G_1(x_0-\varepsilon)}{\varepsilon}=\infty. \tag{2}$$

Indeed, if

$$\frac{G_1(x_0+\varepsilon) - G_1(x_0-\varepsilon)}{\varepsilon} \leq M$$

for any $x_0 \in [a,b]$ and any $\varepsilon > 0$ then one can cover the set Supp $G_1 \bigcap [a,b]$ by the union of a countable set of intervals (a_i, b_i) with $\sum_i (b_i - a_i) < \varepsilon$ (which is possible since the Lebesgue measure of Supp $G_1 \bigcap [a,b]$ is zero) and thus deduce that

$$G_1(b) - G_1(a) \leq M\varepsilon$$

for any $\varepsilon > 0$, which would imply that $G_1(b) = G_1(a)$. Let us fix $x_0 \in [a,b]$ satifying the condition (2) and let us consider the value of

$$\int_{-\infty}^{\infty} H_{\varepsilon^2, x_0}(x) dG_1.$$

Since for $| x - x_0 | \leq \varepsilon$

$$H_{\varepsilon^2, x_0}(x) \geq \frac{e^{-1/4}}{2\sqrt{\pi}\varepsilon} ,$$

we get

$$\int_{-\infty}^{\infty} H_{\varepsilon^2, x_0}(x) dG_1 \geq \int_{x_0-\varepsilon}^{x_0+\varepsilon} H_{\varepsilon^2, x_0}(x) dG_1 \geq \frac{e^{-1/4}}{2\sqrt{\pi}\varepsilon} (G_1(x_0+\varepsilon) - G_1(x_0-\varepsilon)),$$

and thus

$$\limsup_{\varepsilon \to +0} \int_{-\infty}^{\infty} H_{\varepsilon^2, x_0}(x) dG_1 = \infty$$

which gives the desired contradiction and shows that $G_1 = 0$.

Therefore, the measure μ is absolutely continuous, $\mu = D(t)dt$ for a non-negative density function $D(t)$. Then it is sufficient to show that $D(t) = M(t)$ almost everywhere. Indeed, it is sufficient to show that

$$\int_{-\infty}^{\infty} H_{y,a}(t) D(t) dt$$

tends to $D(a)$ for y tending to zero for any point a at which $D(t)$ is continuous. If not, one supposes that, say,

$$F(a) := \lim_{y \to 0} \int_{-\infty}^{\infty} H_{y,a}(t) D(t) dt > D(a)$$

(if $F(a) < D(a)$, the argument is the same). Let us choose $\varepsilon > 0$, $\delta > 0$ such that $D(t) \leq F(a) - \varepsilon$ for any $t \in (a - \delta, a + \delta)$. Then we have

$$\int_{-\infty}^{\infty} H_{y,a}(t) D(t) dt = \left(\int_{a-\delta}^{a+\delta} + \int_{-\infty}^{a-\delta} + \int_{a+\delta}^{\infty} \right) H_{y,a}(t) D(t) dt.$$

Since

$$\frac{1}{2\sqrt{\pi y}}\exp(\frac{-(x-a)^2}{4y}) = \frac{1}{2\sqrt{\pi y}}\exp(\frac{-(x-a)^2}{8y})\cdot\exp(\frac{-(x-a)^2}{8y})$$

$$= \sqrt{2}H_{2y,a}(x)\cdot\exp(\frac{-(x-a)^2}{8y}),$$

we see that

$$\mid \int_{-\infty}^{a-\delta} H_{y,a}(t)D(t)dt + \int_{a+\delta}^{\infty} H_{y,a}(t)D(t)dt \mid\leq$$

$$\mid \int_{-\infty}^{\infty}\sqrt{2}H_{2y,a}(t)D(t)dt \mid e^{\frac{-\delta^2}{8y}}$$

and thus tends to zero for y tending to zero. Therefore,

$$F(a) = \lim_{y\to 0}\int_{-\infty}^{\infty} H_{y,a}(t)D(t)dt = \lim_{y\to 0}\int_{a-\delta}^{a+\delta} H_{y,a}(t)D(t)dt \leq$$

$$(F(a)-\varepsilon)\lim_{y\to 0}\int_{a-\delta}^{a+\delta} H_{y,a}(t)dt \leq (F(a)-\varepsilon)\lim_{y\to 0}\int_{-\infty}^{\infty} H_{y,a}(t)dt = F(a)-\varepsilon,$$

which gives a contradiction, and finishes the proof both of the lemma and the theorem. □

Remark 5.1. GRH Theorem 5.1 and GRH Corollary 5.1 gives a partial answer to the following question of Odlyzko (Open Problem 6.2 of [20]):

Do the zeroes of $\zeta_K(s)$ in the critical strip approach the real axis as $n\to\infty$, and if they do, how fast do they so, and how many of them are there?

Remark 5.2. Theorem 5.1 implies that zeta zeroes are asymptotically uniformly distributed for any asymptotically bad family, e.g., for a family of number fields of fixed absolute degree. This is the main result of [15].

5.2 Function field case

In the function field case, the analogue of Theorem 5.1 is also true. In [36] we proved the Asymptotic Explicit Formula which gives the asymptotic distribution law for Frobenius angles for asymptotically exact families of function fields, or, which is the same, the limit distribution law for zeroes of their zeta-functions. Let $\mathcal{K} = \{K_j\}$ be such a family. For a zero ρ of the zeta-function $\zeta_{K_j}(s)$ let $t(\rho)$ be defined by

$$t(\rho) := \frac{\rho-\frac{1}{2}}{i}.$$

Clearly, $t(\rho)$ is a real number (Weil's theorem) defined modulo 2π, and we suppose that $t(\rho)\in(-\pi,\pi]$, which determines it uniquely.

Let

$$\Delta_j := \frac{\pi}{g_j}\sum_{\zeta_{K_j}(\rho)=0}\delta_{t(\rho)},$$

where $\delta_{t(\rho)}$ is, as usual, the Dirac measure supported at $t(\rho)$. Then Δ_j is a measure of total mass 2π on $\mathbb{R}/2\pi\mathbb{Z}$, and Δ_j is symmetric with respect to $t \mapsto -t$. Points of $\mathbb{R}/2\pi\mathbb{Z}$ are given by their representatives in $(-\pi, \pi]$.

Theorem 5.2. *In the function field case, for an asymptotically exact family, in the weak topology on the space of measures on $\mathbb{R}/2\pi\mathbb{Z}$ the limit*

$$\Delta := \lim_{j\to\infty} \Delta_j$$

exists. Moreover, the measure Δ has a continuous density M_ϕ, and the following Asymptotic Explicit Formula holds:

$$M_\phi(t) = \mathrm{Re}(\xi_\phi(\frac{1}{2} + \frac{i}{\log r} t)) = 1 - \sum_{m=1}^{\infty} m\phi_{r^m} h_m(t)$$

for

$$h_m(t) = \frac{r^{m/2}\cos(mt) - 1}{r^m + 1 - 2r^{m/2}\cos(mt)} \ ,$$

which depends only on the family of numbers $\phi = \{\phi_{r^m}\}$ and we have the following Basic Equality:

$$\xi_\phi(\frac{1}{2}) = 1 - \sum_{m=1}^{\infty} \frac{m\phi_{r^m}}{r^{m/2} - 1} = M_\phi(0) \ .\square$$

Remark 5.3. If one applies Asymptotic Explicit Formula to the case of the maximal family $\mathcal{K}$ (i.e., with $\phi_r = \sqrt{r} - 1$ and $\phi_{r^m} = 0$ for $m \geq 2$) over $\mathbb{F}_r$, $r = p^2$, given by the reductions of curves $X_0(n)$ to characteristic p, then, using the Eichler-Shimura relation, one obtains a particular case (modular forms of weight two) of Serre's results on the asymptotic distribution of eigenvalues of Hecke operators (cf. [30], especially Sections 3 and 7), namely:

Proposition 5.1. *Let p be a fixed prime number, and let $X_n \subset [-2, 2]$ for a positive integer n coprime with p be the set of eigenvalues of the operators $T'_p(n) = T_p(n)/\sqrt{p}$, where $T_p(n)$ is the Hecke operator acting on the space of cusp forms of weight 2 and level n. Then for $n \to \infty$ the set X_n becomes equidistributed with respect to the measure*

$$\mu_p = \frac{p+1}{\pi} \cdot \frac{p\sqrt{1 - x^2/4}\, dx}{(p+1)^2 - px^2} .\square$$

5.3 Lowest zeta-zero

Theorem 5.1 makes it possible to prove (under GRH) that the lowest zero of the zeta-function of a global field tends to 1/2. Most probably, this result is known to experts, but we have not found it in the literature (cf. however, [22]). Let us denote this lowest zero by $\rho_0(K) = 1/2 + t_0(K)i$.

GRH Proposition 5.2. *For any family $\{K_i\}$ of global fields*

$$\lim_{g(K_i)\to\infty} t_0(K_i) = 0.$$

Proof. Let us suppose the contrary:

$$\liminf_{g(K_i)\to\infty} t_0(K_i) = \varepsilon > 0.$$

Passing to a subsequence if necessary, one can suppose that there exists a sequence $K_1, K_2, \ldots$ with $t_0(K_j) \geq \varepsilon$ for any j. Passing to a subsequence once again we suppose that our sequence is asymptotically exact and thus we can apply Theorems 5.1 and 5.2. The condition $t_0(K_j) \geq \varepsilon$ implies that the corresponding limit density M_ϕ identically vanishes on the interval $(-\varepsilon, \varepsilon)$, which is impossible since $M''_\phi(0) > 0$ for $\phi \neq 0$, and M_ϕ equals 1 identically for $\phi = 0$ (note also that $M_\phi(0) > 0$ if the deficiency of the family is positive).□

6 Further theory

In this section we discuss some directions of further study of infinite global fields, and more generally, of asymptotically exact families. Most of the following problems look very difficult, but some of them seem to be easier than the others, and we hope to return to them elsewhere.

6.1 Structure of the parameter set

For an infinite global field $\mathcal{K}$ we have defined the sequence $\phi_{\mathcal{K}}$ of parameters:

$$\phi_{\mathcal{K}} = (\phi_{\mathbb{R}}, \phi_{\mathbb{C}}, \phi_2, \ldots)$$

for the number field case, and

$$\phi_{\mathcal{K}} = (\phi_r, \phi_{r^2}, \phi_{r^3}, \ldots)$$

for the function field one. We have shown that these sequences are sometimes nontrivial (i.e., nonzero). Then it is natural to ask about the structure of possible parameters. We define the sets Φ and Φ_r in $\mathbb{R}^\infty$ by

$$\Phi = \{\phi_{\mathcal{K}} :\ \mathcal{K} \text{ an infinite number field}\},$$

$$\Phi_r = \{\phi_{\mathcal{K}} :\ \mathcal{K} \text{ an infinite function field over } \mathbb{F}_r\}.$$

We also introduce the sets $\tilde{\Phi} \subset \mathbb{R}^\infty$ and $\tilde{\Phi}_r \subset \mathbb{R}^\infty$ defined exactly as Φ and Φ_r , but for all asymptotically exact families. Clearly, $\{0\} \in \Phi \subseteq \tilde{\Phi}$, $\{0\} \in \Phi_r \subseteq \tilde{\Phi}_r$.

Above considerations show that neither of these sets reduce to $\{0\}$. However, their structure remains mysterious. Let us put some natural questions on this structure, for brevity, only in the case of Φ; exactly the same questions are equally interesting for the other three sets.

Problem 6.1. *Is Φ closed in some natural topology on $\mathbb{R}^\infty$?*

One can propose a natural class of topologies to consider. Let $a = (a_{\mathbb{R}}, a_{\mathbb{C}}, a_2, \ldots)$ be a sequence of positive real numbers indexed exactly as the sequences ϕ are. Then one can define the weighted spaces $l_{p,a}$ with $1 \leq p \leq \infty$ using the norm

$$\| x \|_{p,a} := (\sum_\alpha a_\alpha \mid x \mid^p)^{1/p}.$$

Our Basic Inequality says that $\Phi \subseteq l_{1,a}$ with an appropriate sequence a depending on the version of the Basic Inequality used.

Problem 6.2. *Is Φ (relatively) compact in some natural topology on $\mathbb{R}^\infty$?*

Problem 6.3. *Does Φ contain a non-empty open set in $\mathbb{R}^\infty$? Is it convex? Is it a restricted cone, i.e., does $\phi_0 \in \Phi$ imply $\mu\phi_0 \in \Phi$ for $\mu \in [0,1]$? Is it true that $\Phi \subset l_\infty$?*

Problem 6.4. *Show that the cardinality of* $\operatorname{Supp}\phi$ *is unbounded on Φ where* $\operatorname{Supp}\phi$ *is the set of indices α with $\phi_\alpha \neq 0$. Does there exist $\phi \in \Phi$ with infinite* $\operatorname{Supp}\phi$*?*

The last question is also related with the Unramified Fontaine-Mazur conjecture, see Subsection 6.3 below, and with the convergence abscissa for $\zeta_\phi(s)$, cf. Remark 4.1.

6.2 The deficiency problem

Since a complete description of the sets $\tilde{\Phi}$ and Φ is, most probably, very difficult, one can be also interested in possible values of the deficiency, which has his own importance:

Problem 6.5. *Does there exist an infinite global field $\mathcal{K}$ (an asymptotically exact family) with zero deficiency $\delta_{\mathcal{K}}$?*

If the answer is positive, one would like to have an explicit construction of such a family. Note that the positive answer is known for function global fields over $\mathbf{F}_r$ with a square r (it is given by appropriate infinite modular function fields).

More generally, one can put the following

Problem 6.6. *Describe the set of possible values of the deficiency for infinite global fields (asymptotically exact families).*

One can also be interested in properties of this set: whether it is closed, convex (i.e., an interval), of positive measure, etc.

At the moment these problems seem to be inaccessible and we would like to put a more modest question concerning amelioration of existing estimates for δ_K.

Problem 6.7. *Produce an example of an infinite global field $\mathcal{K}$ (or of an asymptotically exact family) with the value $\delta_{\mathcal{K}}$ as small as possible.*

The best example known in the number field case is that of [8], with $\delta_{\mathcal{K}} \leq$ 0.141...

In particular one should consider families with $\phi_q \neq 0$ for at least one prime power q. It looks promising to search for ramified towers $\mathcal{K}$ of number fields with good values of $\delta_{\mathcal{K}}$ to replace the class field towers of Section 9 below.

6.3 Unramified Fontaine-Mazur conjecture

Let K be a number field, p a prime, let T be a finite set of primes of K, none above p, and let $G_{K,T}^{(p)} = \mathrm{Gal}(\mathcal{K}_T^{(p)})$ be the Galois group of the maximal algebraic pro-p extension $\mathcal{K}_T^{(p)}$ of K unramified outside T. Then the unramified Fontaine–Mazur conjecture reads

Any continuous representation

$$\rho : G_{K,T}^{(p)} \to \mathrm{GL}_n(\mathbb{Z}_p)$$

has a finite image.

One can consider a *just-infinite* subextension $\mathcal{L}/K$ of $\mathcal{K}_T^{(p)}$, i.e., an extension which contains no proper infinite subextensions of K. Then Unramified Fontaine-Mazur Conjecture is equivalent to the finiteness of the image for any representation

$$\rho : \mathrm{Gal}(\mathcal{L}/K) \to \mathrm{GL}_n(\mathbb{Z}_p)$$

for all just-infinite extensions $\mathcal{L}/K$.

One says that a pro-p group is *torsion-riddled* if all its open subgroups have torsion. N. Boston [1] put forth the following conjecture:

The Galois group $\mathrm{Gal}(\mathcal{L}/K)$ *is torsion-riddled for any just-infinite* $\mathcal{L}$.

This conjecture would imply the unramified Fontaine–Mazur conjecture and it is ultimately connected with the following

Problem 6.8. *Does there exist an infinite number field $\mathcal{K}$ for which the set $S_{\mathcal{K}}$ of prime powers q such that $\phi_q > 0$ is infinite? If so, characterize such families.*

Boston's conjecture would follow from

Conjecture 6.1. *Let $\mathcal{K}$ be an infinite number field which is just-infinite over K. Then $S_{\mathcal{K}}$ is infinite.*

Indeed, if it is the case, there exist in $\mathrm{Gal}(\mathcal{K}/K)$ infinitely many Frobenius elements of finite order.

Let us remark that Ihara (cf. [12], p. 695) conjectured the existence of an unramified extension $\mathcal{K}$ with $\delta_{\mathcal{K}} = 0$ and infinite $S_{\mathcal{K}}$.

6.4 Around the asymptotic explicit formula

One easily sees that almost all GRH results of our paper have their unconditional counterparts, with one notable exception, namely, GRH Theorem 5.1 (and its consequences: GRH Corollary 5.1 and GRH Theorem 5.3). It is but natural to pose

Problem 6.9. *What are unconditional analogues of GRH Theorem 5.1., Corollary 5.1 and GRH Theorem 5.3?*

At the moment we have no approach to this problem.

6.5 Results specific for the function field case and corresponding problems

In the function field case we have some specific results which do not yet have their number field counterparts. Let us discuss some of them.

In this case we can get rather good estimates for the coefficients of zeta-functions. More precisely, let

$$Z_K(t) = \sum_{m=1}^{\infty} D_m t^m$$

for a function field K; one knows that $D_m = D_m(K)$ is the number of positive divisors of degree m. Then we have (cf. [36], Proposition 4.1 and Theorem 4.1)

Theorem 6.1. *Let $\mathcal{K} = \{K_j\}$ be an asymptotically exact family of function fields over $\mathbb{F}_r$, and let*

$$\mu_0 = \sum_{m=1}^{\infty} \frac{m\phi_{r^m}}{r^m - 1} = 1 - \xi_\phi(1).$$

Then for any real $\mu > 0$

$$\lim_{i\to\infty} \frac{D_{[\mu g_i]}}{g_i} = \mu \log \Lambda + \sum_{m=1}^{\infty} \phi_{r^m} \log \frac{\Lambda^m}{\Lambda^m - 1},$$

where $\Lambda = \Lambda(\mu)$ equals r for $\mu \geq \mu_0$, and is defined from the equation

$$\sum_{m=1}^{\infty} \frac{m\phi_{r^m}}{\Lambda^m - 1} = \mu$$

for $\mu \leq \mu_0$.$\square$

Moreover, one can get even more precise result for the ratio D_m/h. Let $h_j = h(K_j)$ be the class number. Then (cf. [36], Theorem 5.1)

Theorem 6.2. *Let $\mathcal{K} = \{K_j\}$ be an asymptotically exact family. Then for any $\varepsilon > 0$ and any m with $m/g \geq \mu_1 + \varepsilon$, we have*

$$\frac{D_m(K_j)}{h_j} = \frac{r^{m-g+1}}{r-1}(1 + o(1))$$

for $j \to \infty$, $o(1)$ being uniform in m. Here $\mu_1 = \mu_1(\phi)$ is defined as the largest of the two roots of the equation

$$\frac{\mu}{2} + \mu \log_r \frac{\mu}{2} + (2-\mu)\log_r(1 - \frac{\mu}{2}) = -2\log_r \zeta_\phi(1).\square$$

Note that the estimate of Theorem 6.2 is much more precise than that of Theorem 6.1 (there we have an exponential $o(1)$ instead of the multiplicative one of Theorem 6.2).

Problem 6.10. *What is the number field counterpart of the above results on the number of positive divisors?*

It does not look likely that Theorem 6.2 has a proper number field analogue. On the other hand, one can hope to obtain an analogue of Theorem 6.1. (Cf. Lemma 7.5 below.)

In [36] we also cosidered the asymptotic behaviour of $w_m(\mathcal{K})$, the number of classes of positive divisors of degree m. More precisely, let $\mathcal{K} = \{K_j\}$ be an asymptotically exact family, and let

$$w(\mathcal{K}, \mu)_{\inf} := \liminf (w_{[\mu g]}(K_j))^{1/g_j},$$

$$w(\mathcal{K}, \mu)_{\sup} := \limsup (w_{[\mu g]}(K_j))^{1/g_j}$$

for a real number $\mu \in (0, 1)$. Clearly,

$$w(\mathcal{K}, \mu)_{\inf} \le w(\mathcal{K}, \mu)_{\sup} \le d(\mathcal{K}, \mu) = \lim \frac{D_{[\mu g]}(K_j)}{h_j}\ .$$

In [36] (Proposition 6.1 and Theorem 6.1, cf. also [38]) we proved that for $\mu \in (0, 1/r)$ one has $w(\mathcal{K}, \mu)_{\inf} = w(\mathcal{K}, \mu)_{\sup} = d(\mathcal{K}, \mu)$ and that for $\mu > 1/r$ the ratio $w(\mathcal{K}, \mu)_{\inf}/d(\mathcal{K}, \mu)$ is bounded from below by $r^{-\phi_r R_r(1-\mu/\phi_r)}$ for any asymptotic upper bound R_r for r-ary linear codes (recall that $R_r(\delta)$ is a decreasing continuous function on $[0, \frac{r-1}{r}]$ with $R_r(0) = 1,\ R_r(\frac{r-1}{r}) = 0$).

Problem 6.11. *Is it true that for any $\mu \in (0, 1)$ one has $w(\mathcal{K}, \mu)_{\inf} = w(\mathcal{K}, \mu)_{\sup} = d(\mathcal{K}, \mu)$?*

The above results use geometric arguments, in particular, the construction of algebraic geometry codes and have no evident number theory counterparts.

Part II

Around the Brauer–Siegel Theorem

Part 1 was devoted to the general theory of infinite global fields and asymptotically exact families. In this Part we are considering a specific parameter of these fields, which we call the Brauer–Siegel ratio.

For an asymptotically exact family $\mathcal{K} = \{K_i\}$ of global fields consider the limits

$$\mathrm{BS}(\mathcal{K}) = \lim_{i\to\infty} \frac{\log h_i R_i}{g_i}$$

and

$$\varkappa(\mathcal{K}) = \lim_{i\to\infty} \frac{\log \varkappa_i}{g_i}.$$

Here h_i is the class number, R_i the regulator, and $\varkappa_i$ the zeta-residue at $s = 1$. We are going to show that these limits exist and depend only on the numbers $\phi = \{\phi_\alpha\}$. Therefore, $\mathrm{BS}(\mathcal{K})$ is well defined for an infinite global field $\mathcal{K}$.

Let us start with the function field case. It was treated in our papers [34] and [36], therefore we do not present any proofs here. We set $R = 1$ and, of course, ϕ_α can be nonzero only for $\alpha = r^m$, $m = 1, 2, 3, \ldots$.

First of all we have (cf. [34], Corollary 2) the following Generalized Brauer–Siegel Theorem:

For an asymptotically exact family of function fields over $\mathbb{F}_r$ we have

$$\mathrm{BS}(\mathcal{K}) = \lim_{i\to\infty} \frac{\log_r h_i}{g_i} = 1 + \sum_{m=1}^{\infty} \phi_{r^m} \log_r \frac{r^m}{r^m - 1}.$$

We get the function field case analogue of the classical Brauer–Siegel theorem in the asymptotically bad case (i.e., $\phi_\alpha = 0$ for any α). Then $\mathrm{BS}(\mathcal{K}) = 1$.

Next (cf. [34], theorem 5; [36], theorem 3.1) we have the following Bounds:

For any family of function fields over $\mathbb{F}_r$ we have

$$1 \le \liminf_{i\to\infty} \frac{\log_r h_i}{g_i} \le \limsup_{i\to\infty} \frac{\log_r h_i}{g_i} \le 1 + (\sqrt{r} - 1) \log_r \frac{r}{r-1}.$$

We also know some partial existence results. Both bounds 1 and $1 + (\sqrt{r} - 1) \log_r \frac{r}{r-1}$ are attainable. The lower bound 1 is attained for any asymptotically bad family, while $1+(\sqrt{r}-1) \log_r \frac{r}{r-1}$ is attained for any asymptotically maximal family, i.e., such that $\phi_r = \sqrt{r} - 1$ and $\phi_{r^m} = 0$ for all $m \ne 1$. Such families (and even such towers) are known to exist for r being a square.

Another limit parameter $\varkappa(\mathcal{K})$ gives no new information in the function field case, since $\varkappa(\mathcal{K}) = \mathrm{BS}(\mathcal{K}) - 1$.

In what follows we are going to present the number field analogues of these results which happen to be much more complicated.

7 The Generalized Brauer–Siegel Theorem

In this section we prove a generalization of the Brauer–Siegel theorem and present some corollaries.

7.1 Statements

Theorem 7.1 (Generalized Brauer–Siegel Inequality). *For an asymptotically exact family of number fields one has*

$$\limsup_{i\to\infty} \frac{\log(h_i R_i)}{g_i} \le 1 + \sum_q \phi_q \log \frac{q}{q-1} - \phi_{\mathbb{R}} \log 2 - \phi_{\mathbb{C}} \log 2\pi,$$

$$\limsup_{i\to\infty} \frac{\log \varkappa_i}{g_i} \le \sum_q \phi_q \log \frac{q}{q-1},$$

the sum being taken over all prime powers q.

This result is unconditional. Assuming GRH we get the equality.

GRH Theorem 7.2 (GRH Generalized Brauer–Siegel Theorem). *For an asymptotically exact family $\mathcal{K}$ of number fields the limits* $\mathrm{BS}(\mathcal{K})$ *and* $\varkappa(\mathcal{K})$ *exist and we have*

$$\mathrm{BS}(\mathcal{K}) := \lim_{i\to\infty} \frac{\log(h_i R_i)}{g_i} = 1 + \sum_q \phi_q \log\frac{q}{q-1} - \phi_{\mathbb{R}} \log 2 - \phi_{\mathbb{C}} \log 2\pi,$$

$$\varkappa(\mathcal{K}) := \lim_{i\to\infty} \frac{\log \varkappa_i}{g_i} = \sum_q \phi_q \log\frac{q}{q-1},$$

the sum being taken over all prime powers q.

If we restrict our attention to the case of *almost normal towers*, we can prove the same unconditionally. To formulate the result we need one definition more.

Let K be a number field. We call K *almost normal* if there exists a finite tower of number fields $\mathbb{Q} = K_0 \subset K_1 \subset \ldots \subset K_m = K$ such that all the extension K_i/K_{i-1} are normal. A family is called almost normal if all its fields are. An infinite number field is called almost normal if it is a limit of an almost normal tower.

Theorem 7.3 (Unconditional Generalized Brauer–Siegel Theorem). *For an asymptotically good tower $\mathcal{K} = \{K_i\}$, $K_1 \subset K_2 \subset \ldots$, of almost normal number fields (in particular, for an infinite asymptotically good normal number field) the limits* $\mathrm{BS}(\mathcal{K})$ *and* $\varkappa(\mathcal{K})$ *exist and we have*

$$\mathrm{BS}(\mathcal{K}) := \lim_{i\to\infty} \frac{\log(h_i R_i)}{g_i} = 1 + \sum_q \phi_q \log\frac{q}{q-1} - \phi_{\mathbb{R}} \log 2 - \phi_{\mathbb{C}} \log 2\pi,$$

$$\varkappa(\mathcal{K}) := \lim_{i\to\infty} \frac{\log \varkappa_i}{g_i} = \sum_q \phi_q \log\frac{q}{q-1},$$

the sum being taken over all prime powers q.

Remark 7.1. The classical Brauer–Siegel theorem claims that (subject to its conditions) $\varkappa(\mathcal{K}) = 0$. An upper bound for $\varkappa(\mathcal{K})$ was given by Hoffstein [10]. We shall ameliorate on his bound below (Remark 8.2).

7.2 Proofs

Proof of Theorems 7.1, 7.2 *and* 7.3. We begin with the inequality of Theorem 7.1 which does not require additional conditions. Passing to a subfamily we can suppose that there exits the limit (may be, infinite)

$$\lim_{i\to\infty} \frac{\log(h_i R_i)}{g_i}\ .$$

For any real $s > 1$ and any K we have

$$\zeta_K(s) = \frac{\varkappa_K}{s-1} F_K(s)\ ,$$

$\varkappa_K$ being the residue of $\zeta_K(s)$ at 1 and $F_K(s)$ being an analytic function in a neighbourhood of 1 with $F_K(1) = 1$. This is just a way to write the residue.

Let us first remark that

$$\frac{\log \varkappa_{K_j}}{g_j} \longrightarrow \lim_{j\to\infty} \frac{\log(h_j R_j)}{g_j} - 1 + \phi_{\mathbb{R}} \log 2 + \phi_{\mathbb{C}} \log 2\pi \,.$$

To see this, start with the standard formula

$$\varkappa_K = \frac{2^{r_1}(2\pi)^{r_2}(h_K R_K)}{w_K\sqrt{|D_K|}} \,.$$

For $g_j = \log \sqrt{|D_{K_j}|} \to \infty$, i.e., for $j \to \infty$, its logarithm gives exactly what we want, if we note that $\log w_{K_j} / \log |D_{K_j}| \to 0$ since $w_{K_j} \le c n_{K_j}^2$ for an absolute constant c (cf. e.g., [16], proof of Lemma 1 of XVI.1).

Let us put $s = 1 + \theta$ with $\theta = \theta_j > 0$ being a small positive real number, its dependence on j to be specified later. Taking the logarithm of

$$\zeta_{K_j}(s) = \frac{\varkappa_{K_j}}{s-1} F_{K_j}(s)$$

and dividing by g_j we get

$$\frac{\log \zeta_{K_j}(1+\theta_j)}{g_j} = \frac{\log \varkappa_{K_j}}{g_j} + \frac{\log F_{K_j}(1+\theta_j)}{g_j} - \frac{\log \theta_j}{g_j} \,.$$

If $j \to \infty$ then, to prove the theorem, it suffices to show that for a proper choice of θ_j the following three points are satisfied:

(i)

$$\frac{\log \zeta_{K_j}(1+\theta_j)}{g_j} \longrightarrow \sum_q \phi_q \log \frac{q}{q-1};$$

(ii)

$$\frac{\log \theta_j}{g_j} \longrightarrow 0;$$

(iii)

$$\liminf \frac{\log F_{K_j}(1+\theta_j)}{g_j} \ge 0.$$

Let us first look at (i). We have

$$\zeta_{K_j}(1+\theta) = \prod_q (1 - q^{-1-\theta})^{-N_j(q)}$$

for any $\theta > 0$, where $N_j(q)$ is the number of places of K_j with the norm q. Let

$$f_j(\theta) = \frac{\log \zeta_{K_j}(1+\theta)}{g_j}$$

and

$$f(\theta) := \sum_q \phi_q \log \frac{1}{1 - q^{-1-\theta}}.$$

Taking logarithms and dividing by g_j we get

$$f_j(\theta) = \sum_q \frac{N_j(q)}{g_j} \log \frac{1}{1 - q^{-1-\theta}},$$

thus $f_j(\theta) \to f(\theta)$ uniformly for $\theta \geq \theta_0 > 0$ by definition of ϕ_q, $N_j(q)/g_j$ being bounded by an absolute constant and $\sum\limits_q \frac{N_j(q)}{g_j} \log \frac{1}{1-q^{-1-\theta}}$ converging for $\theta > 0$. Moreover,

$$f(\theta) \longrightarrow \sum_q \phi_q \log \frac{q}{q-1} \qquad \text{for} \qquad \theta \to 0 \; ,$$

the series $\sum\limits_q \phi_q \log \frac{q}{q-1}$ being convergent (the series $\sum\limits_q \frac{\phi_q \log q}{q-1}$ which is at most 1 by Proposition 3.2 provides an upper bound for it).

Then we choose a decreasing sequence $\theta(N) > 0$ in such a way that

$$|f(\theta(N)) - \sum_q \phi_q \log \frac{q}{q-1}| < 1/2N \; ,$$

and we can also choose an increasing sequence $j(N)$ such that $g_{j(N)} \geq \frac{1}{\theta(N)}$ and also such that for any $\theta \in [\theta(N+1), \theta(N)]$ we have

$$|f(\theta) - f_{j(N)}(\theta)| < 1/2N.$$

This is possible since $g_j \to \infty$ and $f_j(\theta) \to f(\theta)$ uniformly for $\theta \geq \theta(N+1)$. Then let $N = N(j)$ be given by $j(N) \leq j \leq j(N+1) - 1$ and put $\theta_j = \theta(N(j))$; note that $N(j) \to \infty$. We see that

$$|f_j(\theta_j) - \sum_q \phi_q \log \frac{q}{q-1}| < 1/N(j) \quad \longrightarrow \quad 0$$

which proves (i). We also get (ii) for granted since $1/g_j \leq 1/g_{j(N)} \leq \theta_j$ and hence $\log \theta_j / g_j \to 0$.

For (iii), keeping in mind that $\left(\frac{\log \zeta_{K_j}(s)}{g_j}\right)' = \sum_P \sum_{m=1}^{\infty} r^{-ms} \log r$, we rewrite Stark's formula

$$\log |D| = r_1(\log \pi - \psi(s/2)) + 2r_2(\log(2\pi) - \psi(s)) - \frac{2}{s} - \frac{2}{s-1}$$

$$+2{\sum_\rho}' \frac{1}{s-\rho} + 2 \sum_P \sum_{m=1}^{\infty} r^{-ms} \log r,$$

for $s = 1 + \theta$ as

$$\left(\frac{\log \zeta_{K_j}(1+\theta) + \log \theta}{g_j}\right)' = -1 + \frac{r_1}{2g_j}(\log \pi - \psi(\frac{1+\theta}{2}))$$

$$+\frac{r_2}{g_j}(\log 2\pi - \psi(1+\theta)) - \frac{1}{(1+\theta)g_j} + \sum_\rho{}' \frac{1}{(1+\theta-\rho)g_j},$$

which shows that the derivative

$$\left(\frac{\log F_{K_j}(1+\theta)}{g_j}\right)' = \left(\frac{\log \zeta_j(1+\theta) + \log\theta}{g_j}\right)'$$

is bounded from below by -2 for any small enough θ since all the terms except -1 and $-\frac{1}{(1+\theta)g_j}$ are positive. Thus

$$\frac{\log F_{K_j}(1+\theta_j)}{g_j} \geq c\theta_j \longrightarrow 0\ ,$$

which proves (iii). Summing up, we get an unconditional proof of the desired inequality

$$\lim_{i\to\infty} \frac{\log(\varkappa_i)}{g_i} \leq \sum_q \phi_q \log\frac{q}{q-1}$$

and the corresponding one for $\lim \frac{\log(h_i R_i)}{g_i}$, i.e., that of Theorem 7.1.

To prove Theorem 7.2 one supposes GRH. In fact, it is sufficient to prove that

(iii)$'$

$$\liminf \frac{\log F_{K_j}(1+\theta_j)}{g_j} \leq 0.$$

To do this we shall use the following GRH lemma.

GRH Lemma 7.1. *For any asymptotically exact family of number fields the function*

$$Z_j(s) := \frac{-(\log(\zeta_{K_j}(s))' - 1/(s-1)}{g_j}$$

tends for $j \to \infty$ to

$$Z_\phi(s) := \sum_q \phi_q \frac{\log q}{q^s - 1}$$

uniformly on $\mathrm{Re}(s) \geq 1/2 + \delta$ *for any* $\delta > 0$.

In fact, Lemma 7.1 is the key lemma of Ihara's paper ([12], p. 698), where it is proved in the special case of an unramified tower; his proof stays mostly valid in our situation as well, nevertheless we present it here.

Proof of Lemma 7.1. Note first of all, that the series defining $Z_\phi(s)$ converges uniformly on $\mathrm{Re}\, s \geq 1$. Indeed, it is bounded from above by

$$\sum_q \phi_q \frac{\log q}{q-1} \leq 1.$$

If one assumes GRH, the series becomes uniformly convergent and hence analytic on $\mathrm{Re}\, s > 1/2$, since it is the case for

$$\sum_q \phi_q \frac{\log q}{\sqrt{q}-1} \leq 1.$$

Let us consider a presentation of $Z_j(s)$ and Z_ϕ as Mellin transforms of Chebyshev step functions. We have a well-known and easy to prove formula (cf. [12], eq. 5-5 and 5-6) valid for $\operatorname{Re} s > 1$:

$$s^{-1}Z_j(s) = \frac{1}{g_j}\int_1^\infty (G_j(x) - x)x^{-s-1}dx \ ,$$

where

$$G_j(x) := \sum_{\substack{P,\ m\geq 1\\ N(P)^m\leq x}} \log N(P) = \sum_{\substack{q,\ m\geq 1\\ q^m\leq x}} N_q(K_j)\log q$$

is the Chebyshev step function for the field K_j, and the first sum is taken over all prime divisors P of the field K_j.

Similarly, for Z_ϕ we get for $\operatorname{Re}(s) > 1$

$$s^{-1}Z_\phi(s) = \int_1^\infty H(x)x^{-s-1}dx,$$

where $H(x)$ is an asymptotic analogue of $G_j(x)$:

$$H(x) = \sum_{\substack{q,\ m\geq 1\\ q^m\leq x}} \phi_q \log q.$$

Now we use the Lagarias-Odlyzko estimate for $G_j(s)$ (which uses GRH, cf. [14], Theorem 9.1):

$$|G_j(x) - x| \leq C(n_j\sqrt{x}(\log x)^2 + 2g_j\sqrt{x}\log x) \ ,$$

where $n_j = [K_j : \mathbb{Q}]$ and C is an absolute constant. Thus

$$|G_j(x) - x| \leq C_1 g_j\sqrt{x}(\log x)^2$$

with another absolute constant C_1.

The last formula shows that the integral in the integral representation of $s^{-1}Z_j(s)$ converges for $\operatorname{Re}(s) > 1/2$, and thus the representation is valid for $\operatorname{Re}(s) > 1/2$. The same is true for $s^{-1}Z_\phi(s)$ since it is analytic for $\operatorname{Re}(s) > 1/2$ as explained above. Therefore, for $\operatorname{Re}(s) > 1/2$ we get

$$s^{-1}Z_j(s) - s^{-1}Z_\phi(s) = \int_1^\infty \left(\frac{G_j(x) - x}{g_j} - H(x)\right) x^{-s-1}dx \ .$$

Fix $\delta > 0$. Let then $\operatorname{Re}(s) \geq 1/2 + \delta$ and let $\varepsilon > 0$. We choose $M > 1$ so that

$$C_1\int_M^\infty (\log x)^2 x^{-1-\delta}dx \leq \varepsilon$$

and

$$\int_M^\infty H(x)x^{-\frac{3}{2}-\delta}dx \leq \varepsilon \ .$$

Let then choose $j(M)$ in such a way that for $j \geq j(M)$ we have the following two inequalities:

$$\left|\frac{G_j(x)}{g_j} - H(x)\right| \leq \delta\varepsilon \text{ for } 1 \leq x \leq M$$

and

$$\left|\int_1^M \left(\frac{x}{g_j}\right) x^{-s-1} dx\right| \leq \int_1^M \left(\frac{x}{g_j}\right) x^{-\delta-\frac{3}{2}} dx = \frac{M^{\frac{1}{2}-\delta}-1}{g_j(\frac{1}{2}-\delta)} \leq \varepsilon\ ,$$

which is possible since $N_q(K_j)/g_j$ tends to ϕ_q, and since the sums in the definition of $G_j(x)$ and $H(x)$ contain only finite (and bounded from above) number of terms for $x \leq M$. Here, by abuse of notation, we agree to understand $(M^{\frac{1}{2}-\delta}-1)/(\frac{1}{2}-\delta)$ as $\log M$ if $\delta = 1/2$.

We get

$$|s^{-1}Z_j(s) - s^{-1}Z_\phi(s)| \ = \ \left|\int_1^\infty \left(\frac{G_j(x)-x}{g_j} - H(x)\right) x^{-s-1} dx\right| \ =$$

$$\left|\int_1^M \left(\frac{G_j(x)}{g_j} - H(x)\right) x^{-s-1} dx - \int_1^M \left(\frac{x}{g_j}\right) x^{-s-1} dx + \right.$$

$$\left.\int_M^\infty \left(\frac{G_j(x)-x}{g_j} - H(x)\right) x^{-s-1} dx\right| \ \leq$$

$$\delta\varepsilon \int_1^M x^{-\delta-\frac{3}{2}} dx + \int_1^M \left(\frac{x}{g_j}\right) x^{-\delta-\frac{3}{2}} dx + \left|\int_M^\infty \left(\frac{G_j(x)-x}{g_j}\right) x^{-s-1} dx\right| +$$

$$\int_M^\infty H(x) x^{-s-1} dx \leq \frac{\delta\varepsilon}{\delta+\frac{1}{2}} + \left|\frac{M^{\frac{1}{2}-\delta}-1}{g_j(\frac{1}{2}-\delta)}\right| + C_1 \int_M^\infty (\log x)^2 x^{-1-\delta} dx +$$

$$\int_M^\infty H(x) x^{-\frac{3}{2}-\delta} dx \leq 4\varepsilon$$

for $\mathrm{Re}(s) \geq 1/2 + \delta$ and $j \geq j(M)$ which proves the lemma. Note that $j(M)$ depends on δ. $\square$

End of proof of GRH Theorem 7.2. From Lemma 7.1 it follows, in particular, that $Z_j(s)$ tends to $Z_\phi(s)$ for $j \to \infty$ uniformly on $\mathrm{Re}(s) \geq 1$. Therefore for small enough ε, large enough j, and any $\theta > 0$, we have

$$|(\log F_{K_j}(1+\theta)/g_j)'| = |Z_j(1+\theta)| \leq |Z_\phi(1)| + \varepsilon \leq 1,$$

since $|Z_\phi(1)| < 1$ because of Proposition 3.2. Thus

$$\log F_{K_j}(1+\theta_j)/g_j \leq \theta_j.$$

This proves (iii)$'$ and the theorem.$\square$

Remark 7.2. In fact, one notes that Corollary 7.1 remains valid under the assumption that there are no zeta-zeroes with $\mathrm{Re}(s) \geq 1 - \delta$ for arbitrary fixed $\delta > 0$, so that we do not need the full strength of GRH.

To prove the opposite inequality of Theorem 7.3 it is sufficient to show that for an asymptotically good almost normal tower $\{K_j\}$ and for a proper choice of θ_j the following conditions are satisfied:

$$\text{(i)}'' \qquad \liminf\ f_j(\theta_j) \geq \sum_q \phi_q \log \frac{q}{q-1} = f(0) ;$$

$$\text{(ii)}'' \qquad \frac{\log \theta_j}{g_j} \longrightarrow 0 ;$$

$$\text{(iii)}'' \qquad \limsup \log F_{K_j}(1+\theta_j)/g_j \leq 0 .$$

We should stress that the choice of θ_j in the proof of Theorem 7.3 is completely different from that in the proofs of Theorems 7.1 and 7.2. We need the following definition.

Let K be a number field (of a finite degree n). A real number ρ is called an *exceptional zero* of $\zeta_K(s)$ if $\zeta_K(\rho) = 0$ and

$$1 - (4 \log |D_K|)^{-1} \leq \rho < 1;$$

an exceptional zero ρ of $\zeta_K(s)$ is called its *Siegel zero* if

$$1 - (16 \log |D_K|)^{-1} \leq \rho < 1.$$

It is known that for any K there exists at most one exceptional zero of $\zeta_K(s)$.

We are going to show that under some conditions asymptotically exact families have no Siegel zero. We begin with the following fundamental property of Siegel zeroes discovered by Heilbronn [9] and precised by Stark ([32], Lemma 10):

Lemma 7.2. *Let K be an almost normal number field, and let ρ be a Siegel zero of $\zeta_K(s)$. Then there is a quadratic subfield k of K such that $\zeta_k(\rho) = 0$.* □

Lemma 7.3. *Let $\mathcal{K} = \{K_i\}$ be an asymptotically good family of almost normal number fields. Then there exists a positive integer I such that $\zeta_{K_i}(s)$ has no Siegel zero for any $i \geq I$. In other words, in such a family almost all fields have no Siegel zero.*

Proof. In view of Lemma 7.2 it is sufficient to prove that the set $Q(\mathcal{K})$ of quadratic fields k contained in at least one of the fields K_i is finite. Indeed, if this is the case, let

$$\beta = \max\{\rho \in \mathbb{R} : \zeta_k(\rho) = 0 \text{ for } k \in Q(\mathcal{K})\}.$$

Since $Q(\mathcal{K})$ is finite, the maximum exists and $\beta < 1$. Now if

$$g_{K_i} > \frac{1}{16(1-\beta)}$$

then $\zeta_{K_i}(s)$ has no Siegel zeroes by Lemma 7.2.

Let us verify the finiteness of $Q(\mathcal{K})$. The ratio n/g is non-increasing in extensions, since

$$|D_K| \geq |D_k|^{[K:k]}.$$

Moreover, $n_{K_i}/g_{K_i} \to \phi_\infty = \phi_{\mathbb{R}} + 2\phi_{\mathbb{C}} > 0$, the family $\mathcal{K}$ being asymptotically good. Therefore, there exists a positive real number ε such that $n_{K_i}/g_{K_i} \geq \varepsilon$ for any i. If $k \in Q(\mathcal{K})$, $k \subseteq K_i$ then

$$2/g_k = n_k/g_k \geq n_{K_i}/g_{K_i} \geq \varepsilon.$$

Therefore, $g_k \leq 2/\varepsilon$ for any $k \in Q(\mathcal{K})$, and $|D_k| \leq e^{4/\varepsilon}$, which implies the finiteness of $Q(\mathcal{K})$. □

Corollary 7.1. *Let K_1 be a number field with infinite Hilbert class field tower $\{K_i\}$. Then almost all fields K_i have no Siegel zero.* □

Note that for any $\theta > 0$

$$\left(\frac{\log F_{K_j}(1+\theta)}{g_j}\right)' = Z_j(1+\theta)$$

where

$$Z_j(s) := \frac{-(\log(\zeta_{K_j}(s))' - 1/(s-1)}{g_j}$$

which follows from the definition of F_{K_j}.

Lemma 7.4. *There exist absolute constants C_0 and $C > 0$, such that for any number field K which has no Siegel zero we have*

$$|Z(1+\theta)| \leq Cg^6$$

for any real $\theta \in (0,1)$ and for any $g > C_0$. Here

$$Z(s) := \frac{-(\log(\zeta_K(s))' - 1/(s-1)}{g}.$$

Proof of Lemma 7.4. We use the above presentation of $Z(s)$ as the Mellin transform of the Chebyshev step function:

$$s^{-1}Z(s) = \frac{1}{g}\int_1^\infty (G(x)-x)x^{-s-1}dx\ ,$$

where

$$G(x) := \sum_{\substack{P,\ m\geq 1\\ N(P)^m \leq x}} \log N(P) = \sum_{\substack{q,\ m\geq 1\\ q^m \leq x}} N_q(K)\log q$$

We use then the (unconditional) Lagarias-Odlyzko estimate for $G(x)$ ([14], Theorem 9.2):

$$|G(x)-x| \leq C_1 x \exp\left(-C_2\sqrt{\frac{\log x}{n}}\right) + \frac{x^\rho}{\rho}$$

for $\log x \geq C_3 n g^2$, where $n = [K : \mathbb{Q}]$, C_1, C_2 and C_3 being positive absolute constants, ρ being an eventual exceptinal zero of K; note that since ρ is not a Siegel zero we can suppose that $1-(16g)^{-1} > \rho \geq 1-(4g)^{-1}$. Under that condition one easily verifies, using that $g \geq C_4 n$ for a positive absolute constant C_4, the condition $x^\rho/\rho = o\left(x \exp\left(-C_2\sqrt{\log x/n}\right)\right)$, and we can suppose that

$$|G(x)-x| \leq C_1 x \exp\left(-C_2\sqrt{\frac{\log x}{n}}\right) .$$

Since $g \geq C_4 n$ for a positive absolute constant C_4 we also have

$$|G(x)-x| \leq C_1 x \exp\left(-C_5\sqrt{\frac{\log x}{g}}\right)$$

for $\log x \geq C_6 g^3$ and positive absolute constants C_5 and C_6. Note that for $\log x \leq C_6 g^3$ we have the following trivial estimate

$$0 \leq G(x) \leq C_7 g x \log x$$

with an absolute constant C_7; indeed

$$G(x) = \sum_{\substack{q,\, m \geq 1 \\ q^m \leq x}} N_q(K) \log q \leq n \sum_{\substack{q,\, m \geq 1 \\ q^m \leq x}} \log q \leq C_7 g x \log x,$$

since $n \leq Cg$ and $\sum_{q^m \leq x} \log q \leq C' x \log x$.

Therefore,

$$\left|\frac{Z(1+\theta)}{1+\theta}\right| = \left|\frac{1}{g}\int_1^\infty (G(x)-x)x^{-2-\theta}dx\ \right| =$$

$$\frac{1}{g}\left|\int_1^{\exp(C_6 g^3)} (G(x)-x)x^{-2-\theta}dx + \int_{\exp(C_6 g^3)}^\infty (G(x)-x)x^{-2-\theta}dx\right| \leq$$

$$(C_7+1)\int_1^{\exp(C_6 g^3)} x^{-1-\theta}\log x dx + \frac{C_1}{g}\int_{\exp(C_6 g^3)}^\infty \exp\left(-C_5\sqrt{\frac{\log x}{g}}\right) x^{-1-\theta}dx.$$

Then we have

$$(C_7+1)\int_1^{\exp(C_6 g^3)} x^{-1-\theta}\log x dx \leq C_6 g^3 \frac{(C_7+1)}{\theta}(1-e^{-\theta C_6 g^3}) \leq (C_7+1)C_6^2 g^6$$

and

$$\frac{C_1}{g}\int_{\exp(C_6 g^3)}^\infty \exp\left(-C_5\sqrt{\frac{\log x}{g}}\right) x^{-1-\theta}dx\ = 2C_1\int_{\exp(g\sqrt{C_6})}^\infty z^{-\theta g \log z - C_5 - 1}\log z dz$$

which can be seen by the change of variables $x = z^{g \log z}$.

Since
$$z^{-\theta g \log z} \leq z^{-\theta g^2 \sqrt{C_6}}$$
and
$$\log z \leq z^{g^{-1} C_6^{-1/2} \log(g\sqrt{C_6})}$$
for $z \geq \exp(g\sqrt{C_6})$ and $g > C_0$, for
$$\alpha(g) = g^2\sqrt{C_6}, \quad \beta(g) = g^{-1} C_6^{-1/2} \log(g\sqrt{C_6})$$
we get the following estimate
$$C_1 \int_{\exp(g\sqrt{C_6})}^{\infty} z^{-\theta g \log z - C_5 - 1} \log z dz \leq 2C_1 \int_{\exp(g\sqrt{C_6})}^{\infty} z^{-\theta\alpha(g) + \beta(g) - C_5 - 1} dz =$$
$$\frac{2C_1}{(\theta\alpha(g) - \beta(g) + C_5)} \exp(-g(\theta\alpha(g) - \beta(g) + C_5))\sqrt{C_6} \leq C_8 \exp(-C_9 g) \leq C_8$$
with positive absolute constants C_8 and C_9, which implies the lemma since $(C_7 + 1)C_6^2 g^6 + C_8 \leq C g^6$. $\square$

Now let us set $\theta_j = g_j^{-7}$, and verify condititions (i)$''$, (ii)$''$ and (iii)$''$ for that choice, which achieves the proof of Theorem 7.3. The condition (ii)$''$ is obvious.

Applying Lemmata 7.3 and 7.4 to any field $K = K_j$ from our tower for large enough j we get
$$\left| \left(\frac{\log F_{K_j}(1+\theta)}{g_j} \right)' \right| = |Z_j(1+\theta)| \leq C g_j^6$$
for any $\theta \in (0, 1)$.

Therefore, recalling that $F_{K_j}(1) = 1$, we get
$$\left| \frac{\log F_{K_j}(1+\theta_j)}{g_j} \right| = \left| \int_0^{\theta_j} \left(\frac{\log F_{K_j}(1+\theta)}{g_j} \right)' d\theta \right| \leq C g_j^6 \theta_j = \frac{C}{g_j},$$
which proves (iii)$''$.

Let us prove inequality (i)$''$. We set
$$f_j(\theta) = f_j^{(1)}(\theta) + f_j^{(2)}(\theta),$$
where
$$f_j^{(1)}(\theta) := \sum_p \frac{N_j(p)}{g_j} \log \frac{1}{1 - p^{-1-\theta}}$$
is the sum over prime p, and
$$f_j^{(2)}(\theta) := \sum_{q=p^m, m\geq 2} \frac{N_j(q)}{g_j} \log \frac{1}{1 - q^{-1-\theta}}.$$

Similarly, we set

$$f_j^{(1)}(\theta) := \sum_p \phi_p \log \frac{1}{1-p^{-1-\theta}},$$

$$f_j^{(2)}(\theta) := \sum_{q=p^m, m\geq 2} \phi_q \log \frac{1}{1-q^{-1-\theta}},$$

$$f(\theta) = f_j^{(1)}(\theta) + f_j^{(2)}(\theta).$$

Since for a prime p and any j one has $\phi_p \leq \frac{N_j(p)}{g_j}$, we get $f_j^{(1)}(\theta) \geq f^{(1)}(\theta)$ for any $\theta > 0$. On the other hand, $f_j^{(2)}(\theta)$ and $f^{(2)}(\theta)$ converge uniformly on $\theta \geq -\delta$ with a positive δ, and thus $f_j^{(2)}(\theta_j)$ tends to $f^{(2)}(0)$ for θ_j tending to zero. We get

$$\liminf\ f_j(\theta_j) = \liminf(f_j^{(1)}(\theta_j)+f_j^{(2)}(\theta_j)) \geq \liminf(f^{(1)}(\theta_j)+f^{(2)}(\theta_j)) = f(0).\square$$

7.3 Lower bounds for regulators

As an application of the Generalized Brauer-Siegel Theorem one obtains a lower bound for regulators of number fields in asymptotically good families, which is better than the general bound obtained by Zimmert [40].

GRH Theorem 7.4. *For an asymptotically good family of number fields*

$$\liminf_{i\to\infty} \frac{\log R_i}{g_i} \geq (\log\sqrt{\pi e} + \gamma/2)\phi_{\mathbb{R}} + (\log 2 + \gamma)\phi_{\mathbb{C}}.$$

Proof. We begin with an estimate for the class numbers of fields in question which could be of independent interest.

Proposition 7.1. *For an asymptotically exact family of number fields*

$$\limsup_{i\to\infty} \frac{\log h_i}{g_i} \leq 1 - (\log 2\sqrt{\pi} + \frac{\gamma+1}{2})\phi_{\mathbb{R}} - (\log 4\pi + \gamma)\phi_{\mathbb{C}} + \sum \phi_q \log\frac{q}{q-1}.$$

Proof of Proposition 7.1. Let $\mathcal{K} = \{K_i\}$ be an asymptotically exact family of number fields and let

$$\zeta_{K_i}(s) = \sum_{n=1}^{\infty} D_n^{(i)} n^{-s}$$

be the corresponding zeta functions. We shall use the following result on the asymptotic behaviour of the coefficients $D_n^{(i)}$:

Lemma 7.5. *Let n_i, $i = 1, 2, \ldots,$ be a sequence of positive integers such that the limit*

$$\nu := \lim_{i\to\infty} \frac{n_i}{g_i}$$

exists. Then

$$\limsup_{i\to\infty}\frac{\log D^{(i)}_{n_i}}{g_i}=\nu\limsup_{i\to\infty}\frac{\log D^{(i)}_{n_i}}{n_i}\le\nu+\sum_q\phi_q\log\frac{q}{q-1},$$

where the sum is taken over all prime powers.

Proof of Lemma 7.5. Indeed, from the Euler product for $\zeta_{K_i}(s)$ we see:

a) the function $n\to D^{(i)}_n$ is multiplicative, i.e., $D^{(i)}_{nn'}=D^{(i)}_n D^{(i)}_{n'}$ for coprime n and n'. In particular,

$$D^{(i)}_n=\prod D^{(i)}_{p_j^{m_j}}$$

$n=\prod p_j^{m_j}$ being the prime factorization of n.

b)

$$D^{(i)}_{p^m}=\sum_{\substack{(b_1,\dots,b_m)\\ b_1+2b_2+\dots+mb_m=m}}\prod_{s=1}^{m}\binom{N_{p^s}(K_i)+b_s-1}{b_s},$$

the sum being taken over all partitions of m, $b_i\in\mathbb{Z}$, $b_i\ge 0$.

Let $n_i=\prod p_j^{m_{ij}}$. Then

$$\log D^{(i)}_{n_i}=\sum_j\log D^{(i)}_{p_j^{m_{ij}}},$$

$$D^{(i)}_{p_j^{m_{ij}}}=\sum_{\substack{(b_1,\dots,b_{m_{ij}})\\ b_1+2b_2+\dots+m_{ij}b_{m_{ij}}=m_{ij}}}\prod_{s=1}^{m_{ij}}\binom{N_{p_j^s}(K_i)+b_s-1}{b_s}.$$

This implies

$$\log D^{(i)}_{p_j^{m_{ij}}}\le\log\mathbf{p}(m_{ij})+\max_{\substack{(b_1,\dots,b_{m_{ij}})\\ b_1+2b_2+\dots+m_{ij}b_{m_{ij}}=m_{ij}}}\left(\sum_{s=1}^{m_{ij}}\log\binom{N_{p_j^s}(K_i)+b_s-1}{b_s}\right),$$

where $\mathbf{p}(x)$ is the partition function. Now the argument of the proof of Lemma 3.4.10 of [35] shows that

$$\log D^{(i)}_{p_j^{m_{ij}}}\le O(\sqrt{g})+\log p_j^{m_{ij}}+\phi_{p_j^{m_{ij}}}\log\frac{p_j^{m_{ij}}}{p_j^{m_{ij}}-1},$$

which proves the lemma.□

End of proof of Proposition 7.1. Zimmert's theorem on twin classes [40] (cf. also [21]) states that for $\lambda_{\mathbb{R}}=\log 2\sqrt{\pi}+\frac{\gamma+1}{2}$, $\lambda_{\mathbb{C}}=\log 4\pi+\gamma$, and any class $\mathcal{C}$ of ideals of a number field k

$$\frac{n_{\inf}(\mathcal{C})+n_{\inf}(\mathcal{C}^*)}{2}\le g-\lambda_{\mathbb{R}}r_1-\lambda_{\mathbb{C}}r_2$$

and thus

$$\frac{\nu_{\inf}(\mathcal{C})+\nu_{\inf}(\mathcal{C}^*)}{2}\le 1-\lambda_{\mathbb{R}}\phi_{\mathbb{R}}-\lambda_{\mathbb{C}}\phi_{\mathbb{C}},$$

where $\nu_{\inf}(\mathcal{C}) = n_{\inf}(\mathcal{C})/g$, $n_{\inf}(\mathcal{C})$ being the minimum norm of an ideal from $\mathcal{C}$, and $\mathcal{C}^*$ being the twin class of the class $\mathcal{C}$ (i.e., the class $\mathcal{D}\mathcal{C}^{-1}$, where $\mathcal{D}$ is the class of the different). This implies that the class number h_i of K_i is not greater than two times the number of "small norm" ideals, i.e., those counted in $D^{(i)}_{n'_i}$ for $n'_i \le \tilde{n}_i = g_i(1 - \lambda_{\mathbb{R}}\phi_{\mathbb{R}} - \lambda_{\mathbb{C}}\phi_{\mathbb{C}}) + o(g_i)$. By Lemma 7.5,

$$\limsup_{i\to\infty} \frac{\log h_i}{g_i} \le \limsup_{i\to\infty} \frac{1}{g_i} \log(2 \sum_{n'_i \le \tilde{n}_i} D^{(i)}_{n'_i}) \le \limsup_{i\to\infty} \frac{1}{g_i} \log(\max_{n'_i \le \tilde{n}_i} D^{(i)}_{n'_i})$$

$$\le 1 - \lambda_{\mathbb{R}}\phi_{\mathbb{R}} - \lambda_{\mathbb{C}}\phi_{\mathbb{C}} + \sum \phi_q \log \frac{q}{q-1}.\square$$

End of proof of Theorem 7.4. To finish the proof one compares the Generalized Brauer-Siegel theorem with Proposition 7.1. $\square$

Passing to an asymptotically exact subfamily we easily deduce

GRH Corollary 7.3. *Let $\mathcal{S}$ be any family of number fields which does not contain an asymptotically bad subfamily. Then there exists a strictly positive $A = A(\mathcal{S})$ such that*

$$R(K) \ge A(\sqrt{\pi} e^{(\gamma+1)/2})^{r_1} (2e^{\gamma})^{r_2}$$

for any $K \in \mathcal{S}$.$\square$

Remark 7.3. Let us recall that the resut of Zimmert (implicit in [40], but easily deduced from the argument therein) in our notation reads

$$\liminf_{i\to\infty} \frac{\log R_i}{g_i} \ge (\log 2 + \gamma)\phi_{\mathbb{R}} + 2\gamma\phi_{\mathbb{C}}.$$

The numerical values of Zimmert's coefficients are $\log 2 + \gamma \approx 1.270\ldots$ and $2\gamma \approx 1.154\ldots$; those of ours being $\log\sqrt{\pi e} + \gamma/2 \approx 1.361\ldots$ and $\log 2 + \gamma \approx 1.270\ldots$, respectively.

Applying the same argument to the case of an asymptotically good tower of almost normal number fields we get an unconditional version of Theorem 7.4:

Theorem 7.5. *For an asymptotically good tower of almost normal number fields*

$$\liminf_{i\to\infty} \frac{\log R_i}{g_i} \ge (\log\sqrt{\pi e} + \gamma/2)\phi_{\mathbb{R}} + (\log 2 + \gamma)\phi_{\mathbb{C}}.\square$$

8 Bounds for the Brauer–Siegel Ratios

8.1 Linear programming problem

To get optimal estimates (on both sides) for the limit points of $\frac{\log(h_i R_i)}{g_i}$ and $\frac{\log(\varkappa_i)}{g_i}$ we come to the following linear programming problem.

Let q run over all prime powers and let there be given two sets of non-negative coefficients $\{a_q\}$ and $\{b_q\}$ as well as non-negative a_0, b_0, a_1, b_1 with the properties that if $a_i = 0$ then $b_i = 0$ for all $i = 0, 1, q$. Suppose that

(1)
$$\frac{m}{n} \geq \frac{a_{p^m}}{a_{p^n}} \ ,$$
for any $m \geq n$; and

(2)
$$\frac{b_{q_1}}{a_{q_1}} \geq \frac{b_{q_2}}{a_{q_2}}$$
for any $q_1 \leq q_2$ such that $a_{q_1} \neq 0, \ a_{q_2} \neq 0$,

(3) if $a_0 \neq 0$ and $a_1 \neq 0$ then $a_1 \geq a_0, \ b_1 \geq b_0$, and
$$\frac{b_0}{a_0} \leq \frac{b_1}{a_1} \ ,$$

(4)
$$\sum_q b_q = \sum_q a_q = \infty \ .$$

Consider the following optimization problem:

Find the maximum and the minimum of
$$F(x) = \sum_q b_q x_q - b_0 x_0 - b_1 x_1$$
under the conditions

(i) for any $i = 0, 1,$ or q
$$x_i \geq 0;$$

(ii) for any prime p
$$\sum_{m=1}^{\infty} m x_{p^m} \leq x_0 + 2x_1;$$

(iii)
$$\sum_q a_q x_q + a_0 x_0 + a_1 x_1 \leq 1;$$

(iv) if for some $i = 0, 1, q$ we have $a_i = 0$ then it is supposed that the corresponding $x_i = 0$.

We consider this problem in two versions, either when for all $i = 0, 1, q$ such that $a_i \neq 0$ the corresponding x_i are variables, or when x_0 and x_1 are fixed and x_q vary. We suppose that either x_0 or x_1 is nonzero, since otherwise $\max F(x) = \min F(x) = 0$.

Proposition 8.1. *If $a_1 \neq 0$ then*
$$\min F(x) = -\frac{b_1}{a_1} \ .$$
If $a_1 = 0$ and $a_0 \neq 0$ then
$$\min F(x) = -\frac{b_0}{a_0} \ .$$

The same problem for fixed x_0 and x_1 has

$$\min_{\substack{x_0,x_1\\ fixed}} F(x) = -b_0x_0 - b_1x_1 \ .$$

Proof. The assertion is almost obvious. Indeed, given a vector x with a nonzero x_q for some q, change it, putting $x_q = 0$. The value of $F(x)$ then diminishes, leaving all the conditions satisfied. Therefore the minimum is attained when $x_q = 0$ for every q. Then we see that the minimum of $-b_0x_0 - b_1x_1$ under the conditions $x_i \geq 0$ and $a_0x_0 + a_1x_1 \leq 1$ is attained for $a_0x_0 + a_1x_1 = 1$ and one of the two x_i being 0, namely, that with the smaller ratio b_i/a_i.□

Proposition 8.2. *Suppose that x_0 and x_1 are fixed. Then*

$$\max_{\substack{x_0,x_1\\ fixed}} F(x) = (x_0 + 2x_1)\left(\sum_{p<p'} b_p + \alpha b_{p'}\right) - b_0x_0 - b_1x_1,$$

where p' and $\alpha \in (0, 1]$ are found from the condition

$$\sum_{p<p'} a_p + \alpha a_{p'} = \frac{1 - a_0x_0 - a_1x_1}{x_0 + 2x_1} \ .$$

Proof. Suppose that x satisfies the requirements (i), (ii), and (iii). Let x' coincide with x in all coordinates except $x_{p^m} \neq 0$ and x_{p^n}, $n \leq m$, and set $x'_{p^m} = x_{p^m} - \varepsilon$ and $x'_{p^n} = x_{p^n} + \varepsilon a_{p^m}/a_{p^n}$. Then, as we pass from x to x', the left hand side of (iii) does not change, that of (ii) can only get less because of (1), and $F(x') \geq F(x)$ because of (2). This proves that we can take

(v)

$$x_{p^m} = 0 \quad \text{for } m > 1 \ ,$$

and (ii) is reduced to

(ii′)

$$x_p \leq x_0 + 2x_1 \ .$$

Now let us deal with

(iii′)

$$\sum_p a_px_p + a_0x_0 + a_1x_1 \leq 1 \ .$$

Suppose again that x satisfies the requirements (v), (i), (ii′), and (iii′). Let x' coincide with x in all coordinates but $x_{p_1} \neq 0$ and x_{p_2}, $p_1 \geq p_2$, and set $x'_{p_1} = x_{p_1} - \varepsilon$ and $x'_{p_2} = x_{p_2} + \varepsilon\frac{a_{p_1}}{a_{p_2}}$. Again, as we pass from x to x', the left hand side of (iii′) does not change, and $F(x') \geq F(x)$ because of (v). Therefore, it is profitable for $F(x)$ to make for small indices p the value of x_p as large as possible, i.e., to set

$$x_p = x_0 + 2x_1 \ .$$

This we can do, until it starts to contradict (iii′).

Summing up, we have proved that there exists a prime p' such that the maximum of $F(x)$ is attained for some x satisfying the conditions:

$$x_q = 0 \quad \text{for} \quad q \neq p;$$

$$x_p = 0 \quad \text{for} \quad p > p';$$

$$x_p = x_0 + 2x_1 \quad \text{for} \quad p < p';$$

$$x_{p'} = \alpha(x_0 + 2x_1)$$

for some $\alpha \in (0, 1]$. Here p' and α are chosen in such a way that (iii$'$) becomes an equality. Then

$$F(x) = (x_0 + 2x_1)\left(\sum_{p<p'} b_p + \alpha b_{p'}\right) - b_0x_0 - b_1x_1 \ ,$$

the condition being

$$(x_0 + 2x_1)\left(\sum_{p<p'} a_p + \alpha a_{p'}\right) + a_0x_0 + a_1x_1 = 1 \ . \ \Box$$

Proposition 8.3. *Suppose that x_0, x_1, and all x_q vary. Then*

$$\max F(x) = \frac{\sum\limits_{p \leq p_0} b_p - b}{\sum\limits_{p \leq p_0} a_p + a} \ ,$$

where
$a = \frac{a_1}{2}$, $b = \frac{b_1}{2}$ if $a_0 = 0$ and $a_1 \neq 0$,
$a = a_0$, $b = b_0$ if $a_0 \neq 0$ and $a_1 = 0$,
and if both $a_0 \neq 0$, $a_1 \neq 0$ we have to compare two possibilities $a = \frac{a_1}{2}$, $b = \frac{b_1}{2}$ and $a = a_0$, $b = b_0$.
Here, for each choice of a and b, we let p' run over all primes such that

$$0 \leq \frac{\sum\limits_{p \leq p'} b_p - b}{\sum\limits_{p \leq p'} a_p + a} \leq \frac{b_{p'}}{a_{p'}}$$

and take p_0 to be the greatest of such p'.

Proof. If $x = (x_0, x_1, x_q)$ is a maximum point (one of) for our problem, then it is also a maximum point for the problem with x_0 and x_1 fixed. Therefore, by Proposition 8.2

$$\max_{\substack{x_0, x_1 \\ fixed}} F(x) = (x_0 + 2x_1)\left(\sum_{p<p'} b_p + \alpha b_{p'}\right) - b_0x_0 - b_1x_1 \ .$$

Here $p' = p'(y)$ and $\alpha = \alpha(y)$ depend on and are uniquely determined by

$$y = \frac{1 - a_0 x_0 - a_1 x_1}{x_0 + 2x_1} \; .$$

Recall that $y \geq 0$ because of (iii).

Our first goal is to prove that for a fixed y the maximum is attained when either $x_0 = 0$, or $x_1 = 0$. If $a_0 = 0$ (or $a_1 = 0$) this follows from (iv), so we can consider the case $a_0 > 0$ (or $a_1 > 0$).

Indeed,

$$x_0 = \frac{1 - (a_1 + 2y)x_1}{y + a_0} \geq 0$$

hence

$$0 \leq x_1 \leq \frac{1}{a_1 + 2y} \; .$$

Substituting x_0 into the expression for

$$\max_{x_0, x_1 \text{fixed}} F(x)$$

we see that for a fixed y it is linear in x_1. Thus the maximum is attained at one of the ends, i.e., either for $x_1 = 0$ or for $x_0 = 0$.

We have to maximize each of them over y which is uniquely determined by $p' = p'(y)$ and $\alpha = \alpha(y)$, so we can maximize first over $\alpha \in (0, 1]$ and then over p'. The expressions being linear in α, the maxima are attained at the end, since $\alpha = 0$ and $\alpha = 1$ do not differ up to a change of p'.

Therefore, either $x_0 = 0$ and

$$\max F(x) = \frac{\sum\limits_{p \leq p'} b_p - \frac{b_1}{2}}{\sum\limits_{p \leq p'} a_p + \frac{a_1}{2}}$$

or $x_1 = 0$ and

$$\max F(x) = \frac{\sum\limits_{p \leq p'} b_p - b_0}{\sum\limits_{p \leq p'} a_p + a_0} \; .$$

The last thing to do is to maximize over p'. Let (a, b) be either (a_0, b_0), or $(\frac{a_1}{2}, \frac{b_1}{2})$. When p' grows, at some point the expression

$$\max F(x) = \frac{\sum\limits_{p \leq p'} b_p - b}{\sum\limits_{p \leq p'} a_p + a} \; .$$

becomes positive because of (4). Then we just use the fact that if a, b, A, B are non-negative and

$$\frac{B}{A} \geq \frac{b}{a}$$

then

$$\frac{b}{a} \leq \frac{B+b}{A+a} \leq \frac{B}{A}\,. \square$$

Remark 8.1. The same result could, of course, be obtained by writing out the dual linear problem.

8.2 Bounds

GRH Theorem 8.1 (GRH Bounds). *For any family of number fields*

$$\mathrm{BS}_{\mathrm{lower}} \leq \liminf_{i\to\infty} \frac{\log(h_i R_i)}{g_i} \leq \limsup_{i\to\infty} \frac{\log(h_i R_i)}{g_i} \leq \mathrm{BS}_{\mathrm{upper}},$$

$$0 \leq \liminf_{i\to\infty} \frac{\log \varkappa_i}{g_i} \leq \limsup_{i\to\infty} \frac{\log \varkappa_i}{g_i} \leq \varkappa_{\mathrm{upper}},$$

where

$$\mathrm{BS}_{\mathrm{lower}} = 1 - \frac{\log 2\pi}{\gamma + \log 8\pi} \approx 0.5165...,$$

$$\mathrm{BS}_{\mathrm{upper}} = 1 + \frac{\log\frac{3}{2} + \log\frac{5}{4} + \log\frac{7}{6}}{\frac{\gamma}{2} + \frac{\pi}{4} + \log 2\sqrt{2\pi} + \frac{\log 2}{\sqrt{2}-1} + \frac{\log 3}{\sqrt{3}-1} + \frac{\log 5}{\sqrt{5}-1} + \frac{\log 7}{\sqrt{7}-1}} \approx 1.0938\ldots,$$

$$\varkappa_{\mathrm{upper}} = \frac{\log 2 + \log\frac{3}{2}}{\frac{\gamma}{2} + \log 2\sqrt{2\pi} + \frac{\log 2}{\sqrt{2}-1} + \frac{\log 3}{\sqrt{3}-1}} \approx 0.2164\ldots,.$$

Moreover, if all the fields in the family are totally real then

$$\liminf_{i\to\infty} \frac{\log(h_i R_i)}{g_i} \geq \mathrm{BS}_{\mathbb{R},\mathrm{lower}}$$

where

$$\mathrm{BS}_{\mathbb{R},\mathrm{lower}} = 1 - \frac{\log 2}{\frac{\gamma}{2} + \frac{\pi}{4} + \log 2\sqrt{2\pi}} \approx 0.7419\ldots,$$

and

$$\liminf_{i\to\infty} \frac{\log(\varkappa_i)}{g_i} \leq \varkappa_{\mathbb{R},\mathrm{upper}}$$

where

$$\varkappa_{\mathbb{R},\mathrm{upper}} = \frac{\log 2 + \log\frac{3}{2}}{\frac{\gamma}{2} + \log 2\sqrt{2\pi} + \frac{\pi}{4} + \frac{\log 2}{\sqrt{2}-1} + \frac{\log 3}{\sqrt{3}-1}} \approx 0.1874\ldots,$$

If all the fields are totally complex then

$$\limsup_{i\to\infty} \frac{\log(h_i R_i)}{g_i} \leq \mathrm{BS}_{\mathbb{C},\mathrm{upper}},$$

where

$$\mathrm{BS}_{\mathbb{C},\mathrm{upper}} = 1 + \frac{\sum\limits_{\substack{p=2\\ prime}}^{13} \log\frac{p}{p-1} - \frac{1}{2}\log 2\pi}{\frac{\gamma}{2} + \log 2\sqrt{2\pi} + \sum\limits_{\substack{p=2\\ prime}}^{13} \frac{\log p}{\sqrt{p}-1}} \approx 1.0764\ldots\ .$$

Proof. Since any family contains an asymptotically exact one (Lemma 2.2), any limit point of the ratio is a limit for some asymptotically exact family, and it is enough to prove the theorem for such families. The Generalized Brauer–Siegel Theorem (GRH Theorem 7.2) gives us the limit value of the ratio in terms of $\phi = \{\phi_\alpha\}$. The Basic Inequality (GRH Theorem 3.1) gives us a restriction. Up to a constant 1 we get an optimization problem of the type described above with

$$b_0 = \log 2 \approx 0.693..., \qquad a_0 = \log 2\sqrt{2\pi} + \tfrac{\pi}{4} + \tfrac{\gamma}{2} \approx 2.686...,$$

$$b_1 = \log 2\pi \approx 1.837..., \qquad a_1 = \log 8\pi + \gamma \approx 3.801...,$$

$$b_q = \log\tfrac{q}{q-1}, \qquad a_q = \tfrac{\log q}{\sqrt{q}-1}.$$

We have to check the conditions (1)—(4) above (see the beginning of Subsection 8.1). For (1) we see that

$$\frac{m}{n} \geq \frac{m(p^{n/2}-1)}{n(p^{m/2}-1)} = \frac{a_{p^m}}{a_{p^n}} \quad \text{for} \ \ n \leq m.$$

To check (2) it is enough to prove that

$$f(x) = \frac{(\sqrt{x}-1)\log\frac{x}{x-1}}{\log x}$$

is decreasing for $x \geq 2$. This is quite straightforward.

As for (3), it is obvious because of the numerical values given.

To prove (4), just note that

$$\sum_q a_q \geq \sum_q b_q \geq -\sum_{\substack{p\\ prime}} \log(1-\frac{1}{p}) = \log\zeta(1) = \infty.$$

So we come to the above optimization problem, where $1 + F(x)$ is the right hand side of the Generalized Brauer–Siegel, (i) corresponds to non-negativity of ϕ_q, $\phi_{\mathbb{R}}$ and $\phi_{\mathbb{C}}$, (ii) is the condition of Lemma 2.4, (iii) is the GRH Basic Inequality, and (iv) is empty since all $a_i \neq 0$.

We can now use Proposition 8.1. If $a_1 \neq 0$ we get

$$\min F(x) = -\frac{b_1}{a_1} = -\frac{\log 2\pi}{\log 8\pi + \gamma} \approx -0.4834...,$$

which gives the value of $\mathrm{BS}_{\mathrm{lower}} = 1 + \min F(x)$. If all the fields are totally real, i.e., $a_1 = 0$, Proposition 8.1 gives

$$\min F(x) = -\frac{b_0}{a_0} = -\frac{\log 2}{\frac{\gamma}{2} + \frac{\pi}{4} + \log 2\sqrt{2\pi}} \approx -0.2580...$$

As for the maxima, Proposition 8.3 gives two possibilities, either

$$\max F(x) = C^0_{p'} = \frac{\sum\limits_{p \le p'} b_p - b_0}{\sum\limits_{p \le p'} a_p + a_0} \ ,$$

or

$$\max F(x) = C^1_{p'} = \frac{\sum\limits_{p \le p'} b_p - \frac{b_1}{2}}{\sum\limits_{p \le p'} a_p + \frac{a_1}{2}} \ ,$$

and in both cases we still have to find out p'. Note that $b_2 = b_0 = \log 2$. The values of $C^0_{p'}$ are easily computable, and we have

$$0 = C^0_2 < C^0_3 < C^0_5 < C^0_7 \approx 0.0938...$$

and

$$C^0_7 > \frac{b_{11}}{a_{11}} \approx 0.092...$$

Doing the same for $C^1_{p'}$'s we get

$$C^1_2 < C^1_3 < C^1_5 < C^1_7 < C^1_{11} < C^1_{13} \approx 0.0764...$$

and

$$\frac{b_{17}}{a_{17}} \approx 0.066...$$

Therefore, by Proposition 8.3, $\mathrm{BS}_{\mathrm{upper}} = C^0_7$ and $\mathrm{BS}_{\mathbb{C},\mathrm{upper}} = C^1_{13}$.

The bounds for $\varkappa$ are obtained in the same way, and we leave details to the reader. □

Theorem 8.2 (Unconditional Upper Bound). *For any family of number fields*

$$\limsup_{i \to \infty} \frac{\log(h_i R_i)}{g_i} \le 1 + \frac{\sum\limits_{\substack{p=3 \\ prime}}^{23} \log \frac{p}{p-1}}{\frac{\gamma}{2} + \frac{1}{2} + \log 2\sqrt{\pi} + 2 \sum\limits_{\substack{p=2 \\ prime}}^{23} \log p \sum\limits_{m=1}^{\infty} \frac{1}{p^m+1}} \approx 1.1588\ldots,$$

$$\limsup_{i \to \infty} \frac{\log \varkappa_i}{g_i} \le \varkappa_{\mathrm{unc,upper}} = 1 + \frac{\sum\limits_{\substack{p=3 \\ prime}}^{5} \log \frac{p}{p-1}}{\frac{\gamma}{2} + \log 2\sqrt{\pi} + 2 \sum\limits_{\substack{p=2 \\ prime}}^{5} \log p \sum\limits_{m=1}^{\infty} \frac{1}{p^m+1}} \approx 0.3151\ldots$$

Moreover, if all the fields are totally complex then

$$\limsup_{i\to\infty} \frac{\log(h_i R_i)}{g_i} \le 1 + \frac{\sum\limits_{\substack{p=2\\ prime}}^{179} \log\frac{p}{p-1} - \log\sqrt{2\pi}}{\frac{\gamma}{2} + \log 2\sqrt{\pi} + 2\sum\limits_{\substack{p=2\\ prime}}^{179} \log p \sum\limits_{m=1}^{\infty} \frac{1}{p^m+1}} \approx 1.0965\ldots,$$

and if all the fields are totally real then

$$\limsup_{i\to\infty} \frac{\log \varkappa_i}{g_i} \le 1 + \frac{\sum\limits_{\substack{p=3\\ prime}}^{5} \log\frac{p}{p-1}}{\frac{\gamma}{2} + \log 2\sqrt{\pi} + \frac{1}{2} + 2\sum\limits_{\substack{p=2\\ prime}}^{5} \log p \sum\limits_{m=1}^{\infty} \frac{1}{p^m+1}} \approx 0.2816\ldots.$$

Proof. Along the same lines as above. This time we use the unconditional Basic Inequality′ (Proposition 3.1) and the Generalized Brauer–Siegel Inequality (Theorem 7.1). We get a maximization problem of the same type with

$$b_0 = \log 2 \approx 0.693..., \qquad a'_0 = \log 2\sqrt{\pi} + \tfrac{1}{2} + \tfrac{\gamma}{2} \approx 2.054...,$$

$$b_1 = \log 2\pi \approx 1.837..., \qquad a'_1 = \log 4\pi + \gamma \approx 3.108...,$$

$$b_q = \log\tfrac{q}{q-1}\,, \qquad a'_q = 2\log q \sum_{m=1}^{\infty} (q^m+1)^{-1}\,.$$

Again, we have to check the conditions (1)—(4). This is done as in the previous proof. The only tedious point to check is (2), and again it is enough to show that

$$f(x) = \frac{\log\frac{x}{x-1}}{2\log x \sum\limits_{m=1}^{\infty} (x^m+1)^{-1}}$$

is decreasing, which is again straightforward.

Using, as above, Proposition 8.3 we have to maximize

$$c^0_{p'} = \frac{\sum\limits_{p\le p'} b_p - b_0}{\sum\limits_{p\le p'} a'_p + a'_0}$$

and

$$c^1_{p'} = \frac{\sum\limits_{p\le p'} b_p - \frac{b_1}{2}}{\sum\limits_{p\le p'} a'_p + \frac{a'_1}{2}}\,.$$

We get

$$0 = c^0_2 < c^0_3 < c^0_5 < \ldots < c^0_{23} \approx 0.1588...,$$

$$\frac{b_{29}}{a'_{29}} \approx 0.150... \ .$$

We also have

$$c_2^1 < c_3^1 < c_5^1 < \ldots < c_{179}^1 \approx 0.0965...,$$

$$\frac{b_{181}}{a'_{181}} \approx 0.0964...$$

Again, we leave $\varkappa$ to the reader.□

Remark 8.2. In [10] it is proved that

$$\varkappa(\mathcal{K}) \leq 0.958 - 1.936\phi_{\mathbb{R}} - 2.936\phi_{\mathbb{C}}.$$

Using our estimates one easily gets

$$\varkappa(\mathcal{K}) \leq 0.946 - 1.936\phi_{\mathbb{R}} - 2.936\phi_{\mathbb{C}},$$

and also

$$\varkappa(\mathcal{K}) \leq 0.654 - 1.343\phi_{\mathbb{R}} - 2.032\phi_{\mathbb{C}},$$

which is always better (for $\phi_{\mathbb{R}}$ and $\phi_{\mathbb{C}}$ allowed by the Odlyzko bound). Still better estimates of $\varkappa(\mathcal{K})$ in terms of $\phi_{\mathbb{R}}$ and $\phi_{\mathbb{C}}$ follow from Proposition 8.2. (Note, however, that in [10] the result is not just asymptotic, but effective.)

On the other side we get

Theorem 8.3 (Unconditional Lower Bound). *If for a tower of almost normal number fields there exists $\alpha > 0$ such that* $\inf \frac{N_i}{g_i} \geq \alpha$, *then*

$$\liminf_{i\to\infty} \frac{\log(h_i R_i)}{g_i} \geq 1 - \frac{\log 2\pi}{\gamma + \log 4\pi} \approx 0.4087...$$

If, in addition, the fields are totally real, then

$$\liminf_{i\to\infty} \frac{\log(h_i R_i)}{g_i} \geq 1 - \frac{2\log 2}{\gamma + 1 + \log 4\pi} \approx 0.6625...$$

Sketch of proof. The same argument as in the proof of GRH Theorem 8.1 is applied. We use the unconditional Generalized Brauer–Siegel Theorem (Theorem 7.3) and Proposition 3.1. □

9 Class field towers

9.1 Infinite Unramified Towers with Splitting Conditions

We are going to present some examples. The main goal of this section is to show that the Brauer–Siegel ratio does not necessarily tend to 1. Because of the Brauer–Siegel theorem, we need families of fields for which $[K : \mathbb{Q}]/g_K$ does not tend to 0. One's thought turns immediately to unramified towers, and the only infinite examples we know are Hilbert class field towers satisfying some extra conditions.

Recall that for a field K its Hilbert class field K_{Hilb} is defined as the maximal unramified abelian extension, and $\mathrm{Gal}(K_{\mathrm{Hilb}}/K) = \mathrm{Cl}_K$. We fix a prime ℓ and consider the maximal unramified abelian ℓ-extension $K_{\mathrm{Hilb},\ell}$ with $\mathrm{Gal}(K_{\mathrm{Hilb},\ell}/K) = \mathrm{Cl}_{\ell,K}$, where $\mathrm{Cl}_{\ell,K}$ is the Sylow ℓ-subgroup of Cl_K. Put $K_0 = K, K_1 = K_{\mathrm{Hilb},\ell}, K_2 = (K_1)_{\mathrm{Hilb},\ell}$, etc. There are two possibilities, either $K_n = K_{n+1} = \dots$ for some n, or all these fields are different, i.e., the tower $K = K_0 \subset K_1 \subset K_2 \subset \dots$ is infinite. The latter is the situation we are looking for.

Note that if K_0 is totally real (respectively, totally complex), such are all fields of the tower.

For a group A, let $d_\ell(A) = \dim_{\mathbb{F}_\ell}(A^{\mathrm{ab}}/\ell)$ denote its ℓ-rank. For a field F let $r_1(F)$ be the number of its real, and $r_2(F)$ the number of its complex places.

Consider a degree ℓ extension of number fields K/k. Set $r_1 = r_1(k)$, $r_2 = r_2(k)$, let r be the number of prime ideals of k ramified in K/k and ρ be the number of real places ramified in K/k (i.e., becoming complex), $\delta_\ell = d_\ell(W_K)$, W_K being the group of roots of 1 lying in K, i.e., $\delta_\ell = 0$ if there is no ℓ-root of 1 in k, and $\delta_\ell = 1$ otherwise. In [18] the following statement is proved.

Proposition 9.1 (J.Martinet). *In the above notation, if*

$$r \geq r_1 + r_2 + \delta_\ell + 2 - \rho + 2\sqrt{\ell(r_1 + r_2 - \rho/2) + \delta_\ell}$$

then K has an infinite unramified Hilbert class field ℓ-tower $K = K_0 \subset K_1 \subset K_2 \subset \dots$. $\square$

Here are the best specimens found in the hunt for small discriminants, obtained with the help of Proposition 9.1.

Corollary 9.1 (J.Martinet). *The fields*

$$\mathbb{Q}(\sqrt{3\cdot 5\cdot 13\cdot 29\cdot 61}),$$

$$\mathbb{Q}(\sqrt{2}, \sqrt{3\cdot 5\cdot 7\cdot 23\cdot 29}),$$

$$\mathbb{Q}(\sqrt{-3\cdot 5\cdot 17\cdot 19}),$$

$$\mathbb{Q}(\cos\frac{2\pi}{11}, \sqrt{2}, \sqrt{-23})$$

have infinite unramified class field 2-towers.

Proof. This is a non-obvious corollary of Proposition 9.1 which is proved in [18], Examples 3.2, 4.2, 5.3 and 6.2. $\square$

We shall also need unramified towers with some extra splitting conditions. To study them, let us fix some notation. Let $C_K = J_K/K^*$ be the idèle class group. Let S_∞ be the set of archimedean places of K. Fix a finite set S of prime ideals of K. In what follows if S is empty we omit S in the corresponding notation. Let $O_{K,S}$ be the ring of S-integers, $U_{K,S} = \prod_{v\in S\cup S_\infty} K_v^* \prod_{v\notin S\cup S_\infty} O_{K_v}^*$ the group of idèle S-units, $E_{K,S} = O_{K,S}^* = U_{K,S} \cap K^*$ the group of S-units in K, $I_{K,S}$ the group of fractional ideals nondivisible by prime ideals of S, I_K^S the group of ideals generated by prime ideals in S (of course, $I_K = I_{K,S} \oplus I_K^S$), $P_{K,S}$ the image of the principle ideal group P_K in $I_{K,S}$ under $P_K \subset I_K \to I_{K,S}$, $I_K \to I_{K,S}$ being the natural projection, $\mathrm{Cl}_{K,S} = I_{K,S}/P_{K,S} = \mathrm{Cl}_K / \mathrm{Im}(I_K^S)$ the group of S-classes, W_K the group of roots of unity lying in K.

Theorem 9.1. *Let $P = \{p_1, \ldots, p_t\}$ and $Q = \{q_1, \ldots, q_r\}$ be disjoint sets of prime ideals of k, and let t_0 be the number of principal ideals in P. Consider a number field K/k of prime degree ℓ, ramified exactly at Q. Let S be the set of prime ideals in K lying over P, $s = |S|$. If*

$$r \geq s - t_0 + r_1 + r_2 + \delta_\ell + 2 - \rho + 2\sqrt{\ell(r_1 + r_2 - \rho/2) + \delta_\ell + s}$$

then the field K has an infinite unramified class field ℓ-tower $K = K_0 \subset K_1 \subset K_2 \subset \ldots$, where S splits completely.

To prove this theorem we need some lemmata.

In fact, all constructions of unramified towers we know are based on the following well-known lemma (cf. [25], [17]).

Lemma 9.1. *If*

$$d_\ell(\mathrm{Cl}_{K,S}) \geq 2 + 2\sqrt{d_\ell(E_{K,S}) + 1}$$

then K has an infinite class field ℓ-tower $K = K_0 \subset K_1 \subset K_2 \subset \ldots$, where S splits completely.

Proof. Let L be the union of all K_i, where $K_0 = K$ and K_i/K_{i-1} is the abelian ℓ-extension corresponding by the class field theory to the ℓ-Sylow subgroup of $\mathrm{Cl}_{K_{i-1},S_{i-1}}$, where S_{i-1} consists of all places lying over S. The places of S split completely in L.

Suppose that L/K is of finite degree, $\mathcal{G} = Gal(L/K)$, let S_L be the set of places of L lying over S. As for any finite ℓ-group, we have (see [25], eq.6)

$$\frac{1}{4}(d_\ell(\mathcal{G}))^2 - d_\ell(\mathcal{G}) < d_\ell(H_2(\mathcal{G}, \mathbb{Z})).$$

Then

$$H_2(\mathcal{G}, \mathbb{Z}) = H^{-3}(\mathcal{G}, \mathbb{Z}) = H^{-1}(\mathcal{G}, C_L)$$

by Tate's fundamental theorem (see [33], section 11.3). We have

$$0 \longrightarrow U_{L,S_L}/E_{L,S_L} \longrightarrow C_L \longrightarrow \mathrm{Cl}_{L,S_L} \longrightarrow 0$$

and the ℓ-Sylow subgroup of Cl_{L,S_L} is trivial, otherwise L would yet have another nontrivial ℓ-extension splitting S_L. Hence

$$d_\ell(H^{-1}(\mathcal{G}, C_L)) = d_\ell(H^{-1}(\mathcal{G}, U_{L,S_L}/E_{L,S_L})).$$

The extension L/K being unramified, U_{L,S_L} is cohomologically trivial, since first each local component $\prod\limits_{w \notin S_L \cup S_{L_\infty}} O_{L_w}^*$ is cohomologically trivial, and next the points of S split and hence $\prod_{w\in S_L\cup S_{L_\infty}} L_w^*$ is cohomologically trivial and by definition

$$U_{L,S}=\prod_{w\in S_L\cup S_{L_\infty}} L_w^* \prod_{w\notin S_L\cup S_{L_\infty}} O_{L_w}^*.$$

Hence

$$H^{-1}(\mathcal{G},U_{L,S_L}/E_{L,S_L})=\hat{H}^0(\mathcal{G},E_{L,S_L})=E_{K,S}/N_{L/K}(E_{L,S_L}).$$

The ℓ-rank of the latter being less than or equal to $d_\ell(E_{K,S})$ we see that

$$\frac{1}{4}(d_\ell(\mathcal{G}))^2-d_\ell(\mathcal{G})<d_\ell(E_{K,S}).$$

On the other hand, $d_\ell(\mathcal{G})=d_\ell(\mathcal{G}^{\mathrm{ab}})=d_\ell(Gal(K_1/K))=d_\ell(\mathrm{Cl}_{K,S})$ and we get

$$\frac{1}{4}(d_\ell(\mathrm{Cl}_{K,S}))^2-d_\ell(\mathrm{Cl}_{K,S})<d_\ell(E_{K,S}),$$

i.e.,

$$d_\ell(\mathrm{Cl}_{K,S})<2+2\sqrt{d_\ell(E_{K,S})+1}.\square$$

Lemma 9.2. *We have*

$$\begin{aligned}d_\ell(E_{K,S})&=r_1(K)+r_2(K)+\delta_\ell(K)-1+s\\&=\ell(r_1+r_2-\rho/2)+\delta_\ell-1+s.\end{aligned}$$

Proof. It is well known that $E_{K,S}=W_K\oplus\mathbb{Z}^{r_1(K)+r_2(K)+s-1}$ (cf.[16], V.1). In our case, $r_1(K)=\ell(r_1-\rho)$, $r_2(K)=\ell r_2+\ell\rho/2$, and $\delta_\ell(K)=\delta_\ell$ since no new ℓ-root can appear in an ℓ-extension. $\square$

Lemma 9.3 (J.Martinet). *We have*

$$d_\ell(\mathrm{Cl}_K)\geq r-r_1-r_2+\rho-\delta_\ell.$$

Proof. This is proved in [18], section 2. $\square$

Lemma 9.4. *We have*

$$d_\ell(\mathrm{Cl}_{K,S})\geq d_\ell(\mathrm{Cl}_K)-s+t_0.$$

Proof. Let P_0 be the set of principal ideals lying in P. Let $\phi: I_K^S\to\mathrm{Cl}_K$ be the composition of natural maps $I_K^S\to I_K$ and $I_K\to\mathrm{Cl}_K$. By definition $\mathrm{Cl}_{K,S}=\mathrm{Cl}_K/\operatorname{Im}\phi$. We have $I_K^S=\prod_{p\in P}(\prod_{w|p} w^{\mathbb{Z}})\simeq\mathbb{Z}^s$. Look at the kernel of ϕ. Since for any $p\in P_0$ there is the relation $\prod_{w|p} w\in\operatorname{Ker}\phi$, we get $\mathrm{rk}_{\mathbb{Z}}\operatorname{Ker}\phi\geq t_0$. Therefore, $\mathrm{rk}_{\mathbb{Z}}\operatorname{Im}\phi\leq s-t_0$. It remains to take d_ℓ. $\square$

Proof of Theorem 9.1. By Lemmata 9.2, 9.3, 9.4 and the inequality of the theorem we get

$$\begin{aligned} d_\ell(\mathrm{Cl}_{K,S}) &\geq d_\ell(\mathrm{Cl}_K) - s + t_0 \\ &\geq r - r_1 - r_2 + \rho - \delta_\ell - s + t_0 \\ \geq s - t_0 + r_1 + r_2 + \delta_\ell + 2 - \rho &+ 2\sqrt{\ell(r_1 + r_2 - \rho/2) + \delta_\ell + s} - r_1 - r_2 + \rho - \delta_\ell - s + t_0 \\ &= 2 + 2\sqrt{\ell(r_1 + r_2 - \rho/2) + \delta_\ell + s} \\ &= 2 + 2\sqrt{d_\ell(E_{K,S}) + 1}. \end{aligned}$$

By Lemma 9.1 this proves the theorem. □

Corollary 9.2. *Let $P = \{p_1, \ldots, p_t\}$ and $Q = \{q_1, \ldots, q_r\}$ be disjoint sets of primes. Consider a quadratic number field $K/\mathbb{Q}$ ramified exactly at Q. Let σ be the number of primes in P that split in K, and $s = t + \sigma$ the total number of prime ideals in K lying over P. Suppose that either K is complex quadratic and*

$$r \geq 3 + \sigma + 2\sqrt{2 + s},$$

or K is real quadratic and

$$r \geq 4 + \sigma + 2\sqrt{3 + s}.$$

Then K has an infinite unramified class field 2-tower totally splitting all prime ideals over P.

Proof. Indeed, here $k = \mathbb{Q}$, $\ell = 2$, $\delta_2 = 1$, $r_1 = 1$, $r_2 = 0$, $\rho = 1$ for the complex quadratic case and 0 for the real quadratic one, $t_0 = t$. □

Here are some numerical examples.

Corollary 9.3. *The field*

$$\mathbb{Q}(\sqrt{11{\cdot}13{\cdot}17{\cdot}19{\cdot}23{\cdot}29{\cdot}31{\cdot}37{\cdot}41{\cdot}43{\cdot}47{\cdot}53{\cdot}59{\cdot}61{\cdot}67})$$

has an infinite unramified class field 2-tower in which nine prime ideals lying over 2, 3, 5, 7 and 71 split completely.

Proof. A straightforward check shows that 2, 3, 5, 7 split in $K/\mathbb{Q}$ and 71 is inert. Then we apply Corollary 9.2. □

Corollary 9.4. *The field*

$$\mathbb{Q}(\sqrt{-13{\cdot}17{\cdot}19{\cdot}23{\cdot}29{\cdot}31{\cdot}37{\cdot}41{\cdot}43{\cdot}47{\cdot}53{\cdot}59{\cdot}61{\cdot}73{\cdot}79})$$

has an infinite unramified class field 2-tower in which ten prime ideals lying over 2, 3, 5, 7 and 11 split completely.

Proof. Along the same lines. □

We shall exploit these examples in Subsection 9.3. We also get

Corollary 9.5 (Y.Ihara [12]). *The field*

$$\mathbb{Q}(\sqrt{-3{\cdot}5{\cdot}7{\cdot}11{\cdot}13{\cdot}17{\cdot}23{\cdot}31})$$

has an infinite unramified class field 2-tower in which two prime ideals lying over 2 split completely. □

9.2 A remark on the deficiency problem

In [39] K.Yamamura writes

" Combining Ihara's remark ([12], sect.14) to Golod-Shafarevich theory (cf.[25]) and Martinet's result ([18]), we easily obtain the following

Theorem. *Let K/k be a cyclic extension of degree p (p: a prime number) of an algebraic number field of finite degree. Let $\mathfrak{S}$ be a given set of finite primes of K. Let r' be the number of those finite primes of k which are ramified in K and none of its extension to K belongs to $\mathfrak{S}$. If*

$$r' \geq r_1 + r_2 + \delta_k^{(p)} + 2 - \rho + 2\sqrt{H + p(r_1 + r_2 - \rho/2) + \delta_k^{(p)}},$$

then K has an infinite $\mathfrak{S}$-decomposing p-class field tower. Here ρ denotes the number of real primes of k which are ramified in K, $r_1 = r_1(k)$, $r_2 = r_2(k)$, and $H = |\mathfrak{S}|$. "

In our notation, $r' = r$, $p = \ell$, $\delta_k^{(p)} = \delta_\ell$, $H = s$, $\mathfrak{S} = S$, and the inequality reads

$$r \geq r_1 + r_2 + \delta_\ell + 2 - \rho + 2\sqrt{\ell(r_1 + r_2 - \rho/2) + \delta_\ell + s}.$$

We should admit that we consider the Yamamura theorem to be not only unproved, but most likely false. To explain this point of view let us prove the following

Proposition 9.3. *If the Yamamura theorem is true, the generalized Riemann hypothesis is false.*

Proof. Let $\ell = 2$, $k = \mathbb{Q}$. Consider the field

$$K = \mathbb{Q}(\sqrt{-13\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37\cdot 41\cdot 61\cdot 101}),$$

We have $g_K \approx 17.16493$, $\rho = r_1 = 0$, $r_2 = 1$, $r = 10$, $\delta_2 = 1$. Let S consist of ten ideals lying over 2, 3, 5, 7, 11 (straightforward calculation shows that these primes split in $K/\mathbb{Q}$). Then the Yamamura theorem gives the infiniteness of the unramified class field 2-tower over K since $10 > 3 + 2\sqrt{2 + 10}$. We have

$$\sum_q \frac{\phi_q \log q}{\sqrt{q} - 1} + \alpha_{\mathbb{R}}\phi_{\mathbb{R}} + \alpha_{\mathbb{C}}\phi_{\mathbb{C}} =$$

$$\frac{1}{g}\left(\gamma + \log 8\pi + \frac{2\log 2}{\sqrt{2} - 1} + \frac{2\log 3}{\sqrt{3} - 1} + \frac{2\log 5}{\sqrt{5} - 1} + \frac{2\log 7}{\sqrt{7} - 1} + \frac{2\log 11}{\sqrt{11} - 1}\right) \approx 1.0013... > 1 .$$

This contradicts GRH Basic Inequality (cf. GRH Theorem 3.1 or [12], 2-2).□

Unfortunately enough, the rest of [39] is derived from the above Yamamura theorem and we have to discard all his examples.

In particular, the smallest known deficiency δ is that of Hajir and Maire [8] with $\delta \leq 0.141\ldots$ (cf. the end of Section 3.1). It has $S = \emptyset$. Ihara's example

$$\mathbb{Q}(\sqrt{-3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 23\cdot 31}),$$

S consisting of two divisors of 2, has $\delta \leq 0.248....$

9.3 Examples

GRH Theorem 9.2. *Consider the Martinet field*

$$K = \mathbb{Q}(\cos\frac{2\pi}{11}, \sqrt{2}, \sqrt{-23})$$

of degree 20 *over* $\mathbb{Q}$*. We have*

$$D_K = 2^{30} 11^{16} 23^{10} \ ,$$

$$g = g(K) = \log\sqrt{|D_K|} \approx 45.2578...$$

This field has an infinite unramified 2*-tower* $\mathcal{K}$*, and we have*

$$1 - \frac{10\log(2\pi)}{g} = \mathrm{BS}_{\mathrm{lower}}(\mathcal{K}) \le \mathrm{BS}(\mathcal{K}) \le \mathrm{BS}_{\mathrm{upper}}(\mathcal{K}),$$

$$0 \le \varkappa(\mathcal{K}) \le \varkappa_{\mathrm{upper}}(\mathcal{K}),$$

where

$$\mathrm{BS}_{\mathrm{upper}}(\mathcal{K}) = \mathrm{BS}_{\mathrm{lower}}(\mathcal{K}) + \frac{(\sqrt{23}-1)\log\frac{23}{22}}{\log 23}(1 - \frac{10(\gamma + \log 8\pi)}{g}),$$

$$\varkappa_{\mathrm{upper}}(\mathcal{K}) = \frac{(\sqrt{23}-1)\log\frac{23}{22}}{\log 23}(1 - \frac{10(\gamma + \log 8\pi)}{g}),$$

i.e., approximately,

$$0.5939\ldots \le \mathrm{BS}(\mathcal{K}) \le 0.6025\ldots,$$

$$0 \le \varkappa(\mathcal{K}) \le 0.0086\ldots.$$

The deficiency $\delta(\mathcal{K})$ *of this tower is at most*

$$1 - \frac{10(\gamma + \log 8\pi)}{g} \approx 0.1601\ldots$$

Proof. Recall first (Corollary 9.1) that this field has an infinite unramified tower.

Let $K_0 = \mathbb{Q}(\cos\frac{2\pi}{11})$, $K_{11} = \mathbb{Q}(\sqrt[11]{1})$, $k = \mathbb{Q}(\cos\frac{2\pi}{11}, \sqrt{2})$, $F_{23} = \mathbf{Q}(\sqrt{-23})$, $F_2 = \mathbb{Q}(\sqrt{2})$. The discriminant of a cyclotomic field is well-known (cf., [16], IV.1), so we have

$$D_{K_{11}} = 11^9.$$

Hence, $D_{K_{11}}/\mathbb{Q}$ and $D_{K_0}/\mathbb{Q}$ are unramified outside of 11. Since 11 is totally ramified in $D_{K_{11}/\mathbb{Q}}$, it is also totally ramified in $K_0/\mathbb{Q}$, therefore, $D_{K_0} = 11^4$. The field K is the composite of K_0, F_2 and F_{23}. We get

$$D_K = D_{K_0}^4 D_{F_2}^{10} D_{F_{23}}^{10} = 11^{16} 2^{30} 23^{10},$$

and derive the above value of g.

The deficiency δ for the tower is at most

$$1 - \frac{10(\gamma + \log 8\pi)}{g} \approx 0.1601...$$

Let us first prove that in $K/\mathbb{Q}$ we have the following decomposition of small primes

v	2	3	5	7	11	13	17	19	23
e_v	2	1	1	1	5	1	1	1	2
f_v	5	10	20	10	4	10	10	20	1
n_v	2	2	1	2	1	2	2	1	10

where e_v is the ramification index, f_v the inertia one, and n_v is the number of places over v.

In $K_{11}/\mathbb{Q}$ only 11 is ramified (totally) and for a prime p the inertia index f_p equals the smallest f such that $p^f \equiv 1 (\operatorname{mod} 11)$. We easily check that $f_3 = f_5 = 5$, $f_2 = f_7 = f_{13} = f_{17} = f_{19} = 10$ and $f_{23} = 1$. Since 2 does not divide 5, K_0 being index 2 subfield of K_{11}, we see that in $K_0/\mathbb{Q}$ all the primes of our table except 11 and 23 are inert, i.e., $f_p = 5$, that 11 is totally ramified and 23 is totally split. In $F_2/\mathbb{Q}$ only 2 is ramified, and p is split if and only if 2 is a square modulo p, i.e., if and only if $p \equiv \pm 1 (\operatorname{mod} 8)$; such primes are 7, 17 and 23, and the rest (3, 5, 11, 13, 19) are inert. In $F_{23}/\mathbb{Q}$ the only ramified prime is 23 since $-23 \equiv 1 (\operatorname{mod} 4)$, and the splitting condition is for -23 to be a square $\operatorname{mod} p$. Thus 2, 3, 13 are split, and 5, 7, 11, 17 and 19 are inert.

Summing up this information we get the above table. Note also that in $k/\mathbb{Q}$ there are 10 prime ideals over 23, and they all ramify in K/k.

Looking at the decomposition table above, we see that the smallest norm for which there exists a prime ideal of K is 23 (indeed, $2^5 > 23$, etc.)

The next thing to do is to apply the linear programming approach of Subsection 8.1 to get the minimum and maximum of $\mathrm{BS}(\mathcal{K})$. We set $a_0 = 0$, as well as $a_q = 0$ for $q < 23$, recalling that if $a_i = 0$ we have also $b_i = 0$ and we do not optimize over x_i. The right-hand side of GRH Theorem 7.2 becomes

$$1 - \frac{10}{g} \log 2\pi + F(x) = \mathrm{BS}_{\mathrm{lower}}(\mathcal{K}) + F(x),$$

where

$$F(x) = \sum_{q \geq 23} b_q x_q, b_q = \log \frac{q}{q-1}, x_q = \phi_q \text{ for } q \geq 23.$$

The restrictions are, as usual, (i) $x_q \geq 0$, (ii) $\sum\limits_{m=1}^{\infty} m x_{p^m} \leq \frac{20}{g}$ for any p, and (iii) $\frac{10}{g}(\gamma + \log 8\pi) + \sum\limits_{q \geq 23} a_q x_q \leq 1$, where $a_q = \frac{\log q}{\sqrt{q}-1}$.

The minimum of $F(x)$ is clearly 0, and it is easy to check that the maximum is attained for all $x_q = 0$ except for

$$x_{23} = \frac{\sqrt{23}-1}{\log 23}(1 - \frac{10}{g}(\gamma + \log 8\pi)).$$

Indeed, $x_{23} \approx 0.19... < \frac{20}{g}$ which checks (ii), and x_{23} is chosen so that (iii) becomes an equality.

As for $\varkappa(\mathcal{K})$, we have

$$\varkappa(\mathcal{K}) = \mathrm{BS}(\mathcal{K}) - 1 + \phi_{\mathbb{C}} \log(2\pi).\square$$

Remark 9.1 In all our examples, once we have some information on $\mathrm{BS}(\mathcal{K})$, we also have it on $\varkappa(\mathcal{K})$, the difference between the two being known. That is why, in many cases, we do not say a word about $\varkappa(\mathcal{K})$.

GRH Theorem 9.3. *Consider the real Martinet field*

$$K = \mathbb{Q}(\sqrt{2}, \sqrt{3\cdot5\cdot7\cdot23\cdot29})$$

of degree 4 *over* $\mathbb{Q}$. *We have*

$$D_K = 2^8\cdot(3\cdot5\cdot7\cdot23\cdot29)^2\ ,$$

$$g = g(K) = \log\sqrt{|D_K|} \approx 13.9293...$$

This field has an infinite unramified 2*-tower* $\mathcal{K}$. *Then*

$$\mathrm{BS}(\mathcal{K}) \in (\mathrm{BS}_{\mathrm{lower}}(\mathcal{K}), \mathrm{BS}_{\mathrm{upper}}(\mathcal{K})),$$

where

$$\mathrm{BS}_{\mathrm{lower}}(\mathcal{K}) = 1 - \frac{4\log 2}{g}$$

and

$$\mathrm{BS}_{\mathrm{upper}}(\mathcal{K}) = \mathrm{BS}_{\mathrm{lower}}(\mathcal{K}) + \frac{\log 2}{g} + \frac{\sqrt{7}-1}{g\log 7}\left(g - 2\gamma - \pi - 2\log 8\pi - \frac{\log 2}{\sqrt{2}-1}\right)\log\frac{7}{6},$$

i.e., approximately in the interval

$$(0.8009...\,, 0.8648...).$$

The deficiency of this tower is at most

$$1 - \frac{2\gamma + \pi + 2\log 8\pi}{g} \approx 0.2286...$$

Proof. We proceed as in the proof of Theorem 9.2. Our field has an infinite unramified tower (Corollary 9.1). Let $K_1 = \mathbb{Q}(\sqrt{2})$, $K_2 = \mathbb{Q}(\sqrt{3\cdot5\cdot7\cdot23\cdot29})$, $K = K_1\cdot K_2$. Since $3\cdot5\cdot7\cdot23\cdot29 = 70035 \equiv 3 (\mathrm{mod}\, 4)$, we have $D_{K_2} = 4\cdot70035$ and 2, 3, 5, 7, 23 and 29 are ramified in K_2. Since $D_K = 2^8\cdot70035^2$, we see that the ideal lying over 2 is also ramified in K/K_2, i.e., 2 is totally ramified in $K/\mathbb{Q}$. In $K_1/\mathbb{Q}$ only 2 is ramified, $D_{K_1} = 8$. Since 2 is congruent to a square modulo 7, and noncongruent to a square modulo 3 and 5, we see that 7 splits, but 3 and 5 remain inert in $K_1/\mathbb{Q}$. Thus in $K/\mathbb{Q}$ there is one ideal of norm 2, no ideals of norm 3 and 5, and two ideals of norm 7. There are 4 real places and no complex ones.

Using, as above, the linear programming approach, with $a_1 = a_3 = a_5 = 0$ and $x_2 \le \frac{1}{g}$, we get

$$\mathrm{BS}_{\mathrm{lower}}(K) = 1 - \frac{4\log 2}{g}$$

and

$$\mathrm{BS}_{\mathrm{upper}}(K) = \mathrm{BS}_{\mathrm{lower}}(K) + \max F(x)\ ,$$

where $F(x) = \sum\limits_{q \neq 3,5} b_q x_q$.

The maximum is attained for $x_2 = \frac{1}{g}$, $x_q = 0$ for $q = 4$ and $q > 7$, and the value of x_7 is chosen so that (iii) becomes an equality. □

Remark 9.2. The other two fields of Corollary 9.1 give the following numerical results. For

$$K = \mathbb{Q}(\sqrt{-3{\cdot}5{\cdot}17{\cdot}19})$$

we GRH–have

$$g(K) \approx 4.9359...,$$

$$\delta(\mathcal{K}) \le 0.2298...,$$

and

$$\mathrm{BS}_{\mathrm{lower}}(\mathcal{K}) \approx 0.6276...,\ \mathrm{BS}_{\mathrm{upper}}(\mathcal{K}) \approx 0.6402...$$

For

$$K = \mathbb{Q}(\sqrt{3{\cdot}5{\cdot}13{\cdot}29{\cdot}61})$$

we GRH–have

$$g(K) \approx 7.0687...,$$

$$\delta(\mathcal{K}) \le 0.2400...,$$

and

$$\mathrm{BS}_{\mathrm{lower}}(\mathcal{K}) \approx 0.8038...,\ \mathrm{BS}_{\mathrm{upper}}(\mathcal{K}) \approx 0.9020...$$

GRH Theorem 9.4. *Consider the totally real quadratic field*

$$K = \mathbb{Q}(\sqrt{11{\cdot}13{\cdot}17{\cdot}19{\cdot}23{\cdot}29{\cdot}31{\cdot}37{\cdot}41{\cdot}43{\cdot}47{\cdot}53{\cdot}59{\cdot}61{\cdot}67}).$$

The genus of this field is $g \approx 25.9882\ldots$. *This field has an infinite unramified* 2-*tower* $\mathcal{K}$ *in which nine prime ideals lying over* 2, 3, 5, 7 *and* 71 *split completely. Then* $\mathrm{BS}(\mathcal{K}) \in (\mathrm{BS}_{\mathrm{lower}}(\mathcal{K}), \mathrm{BS}_{\mathrm{upper}}(\mathcal{K}))$ *and* $\varkappa(\mathcal{K}) \in (\varkappa_{\mathrm{lower}}(\mathcal{K}), \varkappa_{\mathrm{upper}}(\mathcal{K}))$, *where*

$$\mathrm{BS}_{\mathrm{lower}}(\mathcal{K}) = 1 + \frac{2\log\frac{3}{2} + 2\log\frac{5}{4} + 2\log\frac{7}{6} + \log\frac{5041}{5040}}{g},$$

$$\mathrm{BS}_{\mathrm{upper}}(\mathcal{K}) = \mathrm{BS}_{\mathrm{lower}}(\mathcal{K}) + \frac{1}{g}\sum_{p=11}^{47} \log\frac{p}{p-1} +$$

$$\frac{\sqrt{53}-1}{g\log 53}\left(g-\gamma-\frac{\pi}{2}-\log 8\pi-2\sum_{p=2}^{7}\frac{\log p}{\sqrt{p}-1}-\frac{\log 71^2}{70}-\sum_{p=11}^{47}\frac{\log p}{\sqrt{p}-1}\right)\log\frac{53}{52},$$

$$\varkappa_{\mathrm{lower}}(\mathcal{K})=\frac{2\log 2+2\log\frac{3}{2}+2\log\frac{5}{4}+2\log\frac{7}{6}+\log\frac{5041}{5040}}{g},$$

$$\varkappa_{\mathrm{upper}}(\mathcal{K})=\mathrm{BS}_{\mathrm{upper}}(\mathcal{K})-1+\frac{2\log 2}{g},$$

the sums being taken over prime p's. Numerically

$$\mathrm{BS}(\mathcal{K})\in(1.0602\ldots,1.0798\ldots),$$

$$\varkappa(\mathcal{K})\in(0.1135\ldots,0.1331\ldots).$$

Proof. Let $d=11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47\cdot 53\cdot 59\cdot 61\cdot 67$. An easy, though tedious check shows that d is congruent to 1 modulo 8, and it is a square modulo 3, 5 and 7, but not modulo 71. Hence 2, 3, 5, 7 split in $K/\mathbb{Q}$ and 71 is inert. Corollary 9.3 shows that there is an unramified tower splitting the nine ideals over 2, 3, 5, 7 and 71. Then we use the same linear programming approach of Subsection 7.1 to calculate $\mathrm{BS}_{\mathrm{lower}}(K)$, $\mathrm{BS}_{\mathrm{upper}}(K)$, $\varkappa_{\mathrm{lower}}(\mathcal{K})$ and $\varkappa_{\mathrm{upper}}(\mathcal{K})$.□

GRH Theorem 9.5. *Consider the totally complex quadratic field*

$$K=\mathbb{Q}(\sqrt{-13\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47\cdot 53\cdot 59\cdot 61\cdot 73\cdot 79}).$$

This field has an infinite unramified 2-tower $\mathcal{K}$ in which ten prime ideals lying over 2, 3, 5, 7 and 11 split comletely. The genus of this field is $g\approx 27.0169...$ Then $\mathrm{BS}(\mathcal{K})\in(\mathrm{BS}_{\mathrm{lower}}(\mathcal{K}),\mathrm{BS}_{\mathrm{upper}}(\mathcal{K}))$ and $\varkappa(\mathcal{K})\in(\varkappa_{\mathrm{lower}}(\mathcal{K}),\varkappa_{\mathrm{upper}}(\mathcal{K}))$, where

$$\mathrm{BS}_{\mathrm{lower}}(K)=1-\frac{1}{g}\log 2\pi+\frac{2}{g}\left(\log 2+\log(3/2)+\log(5/4)+\log(7/6)+\log(11/10)\right),$$

$$\mathrm{BS}_{\mathrm{upper}}(\mathcal{K})=\mathrm{BS}_{\mathrm{lower}}(\mathcal{K})+\frac{1}{g}\sum_{p=13}^{61}\log\frac{p}{p-1}$$

$$+\frac{\sqrt{67}-1}{g\log 67}\left(g-\gamma-\log 8\pi-2\sum_{p=2}^{11}\frac{\log p}{\sqrt{p}-1}-\sum_{p=13}^{61}\frac{\log p}{\sqrt{p}-1}\right)\log\frac{67}{66}\quad,$$

$$\varkappa_{\mathrm{lower}}(K)=\frac{2}{g}\left(\log 2+\log(3/2)+\log(5/4)+\log(7/6)+\log(11/10)\right),$$

$$\varkappa_{\mathrm{upper}}(K)=\mathrm{BS}_{\mathrm{upper}}(\mathcal{K})-1+\frac{1}{g}\log 2\pi,$$

the sums being taken over prime p. Numerically

$$(\mathrm{BS}_{\mathrm{lower}}(\mathcal{K}),\mathrm{BS}_{\mathrm{upper}}(\mathcal{K}))=(1.0482...,1.0653...),$$

$$(\varkappa_{\text{lower}}(\mathcal{K}), \varkappa_{\text{upper}}(\mathcal{K})) = (0.1162..., 0.1333...).$$

Proof. Along the same lines as the proofs of Theorems 9.4, 9.3 and 9.2, using Corollary 9.4. □

Remark 9.3. Ihara's example of Corollary 9.5

$$K = \mathbb{Q}(\sqrt{-3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot23\cdot31})$$

with two divisors of 2 splitting in the tower, has $g \approx 9.5097...$, its deficiency δ is at most 0.2483... and $(\mathrm{BS}_{\text{lower}}(\mathcal{K}), \mathrm{BS}_{\text{upper}}(\mathcal{K})) = (0.9525..., 1.010...)$.

Let us see what can be got without GRH. We consider the same fields as in GRH Theorems 9.2 and 9.3.

Theorem 9.7. *Consider the Martinet field*

$$K = \mathbb{Q}(\cos\frac{2\pi}{11}, \sqrt{2}, \sqrt{-23})$$

of degree 20 *over* $\mathbb{Q}$. *This field has an infinite unramified* 2*-tower* $\mathcal{K}$, *and* $\mathrm{BS}(\mathcal{K}) \in (\mathrm{BS}_{\text{lower}}(\mathcal{K}), \mathrm{BS}_{\text{unc,upper}}(\mathcal{K}))$, *where*

$$\mathrm{BS}_{\text{lower}}(\mathcal{K}) = 1 - \frac{10\log(2\pi)}{g}$$

and

$$\mathrm{BS}_{\text{unc,upper}}(\mathcal{K}) = \mathrm{BS}_{\text{lower}}(\mathcal{K}) + \frac{10\log(\frac{23}{22})}{g} + \frac{2\log(\frac{32}{31})}{g} + \frac{20}{g}\sum_{p=37}^{97}\log\frac{p}{p-1},$$

i.e., approximately in the interval

$$(0.5939..., 0.7108...).$$

Proof. We proceed along the same lines as before. Having no GRH at hand, instead of GRH Theorem 7.2 we use Theorem 7.3 (the tower being almost normal, as any 2-tower over a normal field), and instead of GRH Theorem 3.1 we use either Proposition 3.1 or Proposition 3.2. The latter is easier to calculate. Knowing the decomposition law for small primes (cf. the proof of GRH Theorem 9.2), we see that in K there are 10 infinite complex places, 10 places whose norm is 23, 2 places of norm 32 and no other places of norm strictly smaller than 37. Over any other prime there are at most 20 places.

Then we use the optimization procedure of Section 8, that shows that to get an upper bound we can exaggerate the number of places with small norms. Suppose that there were 20 ideals of each of the norms from 37 to 97 (in fact, there are much less). Even this would contradict the inequality of Proposition 3.2, i.e.,

$$\frac{10}{g}(\gamma + \log 2\pi) + \frac{10}{g}\frac{\log 23}{22} + \frac{2}{g}\frac{\log 32}{31} + \frac{20}{g}\sum_{p=37}^{97}\frac{\log p}{p-1} > 1\ ,$$

the sum being taken over primes. Therefore, by the inequality of Theorem 7.1, any limit point of the Brauer–Siegel ratio is at most

$$1-\frac{10}{g}\log 2\pi+\frac{10}{g}\log\frac{23}{22}+\frac{2}{g}\log\frac{32}{31}+\frac{20}{g}\sum_{p=37}^{97}\log\frac{p}{p-1}\approx 0.7108...\ \square$$

Remark 9.4. Using Proposition 3.1 instead of Proposition 3.2 we can do better. We can also use further information on prime decomposition in $K/\mathbb{Q}$. (In particular, the only possible norms between 37 and 1000 are in fact 121, 353, 439, 463, 593, 967 and 991.) This makes the constant better. Namely, we can prove that $\mathrm{BS}(\mathcal{K})\le 0.623\ldots$

Theorem 9.8. *Consider the real Martinet field*

$$K=\mathbb{Q}(\sqrt{2},\sqrt{3\cdot 5\cdot 7\cdot 23\cdot 29})$$

of degree 4 *over* $\mathbb{Q}$. *Its genus equals*

$$g=g(K)=\log\sqrt{|D_K|}\approx 13.9293...,$$

it has an infinite unramified 2*-tower* $\mathcal{K}$*, and* $\mathrm{BS}(\mathcal{K})\in(\mathrm{BS}_{\mathrm{lower}}(\mathcal{K}),\mathrm{BS}_{\mathrm{unc,upper}}(\mathcal{K}))$*, where*

$$\mathrm{BS}_{\mathrm{unc,upper}}(\mathcal{K})=\mathrm{BS}_{\mathrm{lower}}(\mathcal{K})+\frac{\log 2+2\log\frac{7}{6}+4\log\frac{11}{10}+4\log\frac{13}{12}}{g}+$$

$$\frac{1}{2gA_{17}}\left(g-2\gamma-\pi-2\log 8\pi-A_2-2A_7-4A_{11}-4A_{13}\right)\log\frac{17}{16}\,,$$

where $A_p=2\log p\sum_{m=1}^{\infty}(p^m+1)^{-1}$*, i.e., approximately in the interval*

$$(0.8009\ldots,0.9248\ldots).\square$$

Remark 9.4. Applying the same technique to the fields of GRH Theorems 9.4 and 9.5 we get the results presented in the table at the end of Section 1. We do not write out here the exact formulae which are rather cumbersome.

10 Open questions

In this section we discuss some open questions concerning the Generalized Brauer–Siegel Theorem. First of all, in the proof of the Generalized Brauer–Siegel Theorem we do not really use the whole strength of GRH; moreover, under some mild conditions we have totally dispensed with GRH (Theorem 7.3). Therefore, it is but natural to ask whether one really needs GRH to prove the result, which leads to

Problem 10.1. *Prove GRH Theorem* 7.2 *unconditionally.*

Let us now discuss some problems, connected with the Brauer–Siegel ratio introduced and studied above. First of all, its very existence (i.e., the existence of the corresponding limit) is proved only under GRH or for almost normal asymptotically good infinite global fields, which leads to

Problem 10.2. *Prove unconditionally that for any asymptotically exact family $\mathcal{K}$ of number fields the Brauer–Siegel ratio* $\mathrm{BS}(\mathcal{K})$ *is well defined.*

The following problem is connected with the fact that for an arbitrary asymptotically exact family unconditionally we have only an *upper* bound for $\mathrm{BS}(\mathcal{K})$, cf. Theorems 7.1, 7.3 and 8.2.

Problem 10.3. *Give an unconditional lower bound for the Brauer–Siegel ratio* $\mathrm{BS}(\mathcal{K})$ *for* any *asymptotically exact family.*

Note that for towers of normal number fields this results from Theorems 7.3 and 8.2. One can hope that this problem can be solved if one ameliorates the technique of the usual proof of the Brauer–Siegel theorem, i.e., estimates in the (adelic) integral representation of the zeta-function (cf. Lemma 3 of Section XVI.2 of [16]).

There also is the question of how good our bounds and examples are.

Problem 10.4. *Ameliorate on the bounds of GRH Theorem 8.1 and/or of Theorem 8.2.*

Problem 10.5. *Construct examples of class field towers* (*or other asymptotically exact families*) *with* $\mathrm{BS}(\mathcal{K})$ *GRH–smaller than those of GRH Theorems 6.2 and 9.3 or GRH–greater than those of GRH Theorems 9.4 and 9.5.*

Our results in the present paper are of an asymptotic nature. However, it is clear that a good part of them can be made effective which leads to

Problem 10.6. *Give effective versions of the above results with the remainder terms as good as possible.*

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