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SOME REMARKS ON THE *abc*-CONJECTURE

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ABSTRACT. Let $r(x)$ be the product of all distinct primes dividing a nonzero integer x. The abc-conjecture says that if a, b, c are nonzero relatively prime integers such that $a+b+c=0$, then the biggest limit point of the numbers

$$\frac{\log \max(|a|, |b|, |c|)}{\log r(abc)}$$

equals 1. We show that in a natural anologue of this conjecture for $n \geq 3$ integers, the largest limit point should be replaced by at least $2n-5$. We present an algorithm leading to numerous examples of triples a, b, c for which the above quotients strongly deviate from the conjectural value 1.

1. INTRODUCTION

Let a, b, c be nonzero integers such that

$$a+b+c=0 \quad \text{and} \quad \gcd(a, b, c)=1,$$

and let $r(abc)$ be the product of distinct prime numbers dividing abc. J. Oesterlé posed the question whether the numbers

$$L = L(a, b, c) = \frac{\log \max(|a|, |b|, |c|)}{\log r(abc)} \tag{1}$$

are bounded. This question was refined by D. W. Masser who conjectured that for each $\varepsilon > 0$ there exists a positive constant $C(\varepsilon)$ such that

$$\max(|a|, |b|, |c|) \leq C(\varepsilon) r(abc)^{1+\varepsilon}.$$

This is the abc-conjecture. It is easy to see that the abc-conjecture is equivalent to the inequality

$$\limsup\{L\} \leq 1,$$

where $\limsup\{L\}$ denotes the largest limit point of the quotients (1). But it is not difficult to show that there is a limit point of this set which is ≥ 1. Thus the abc-conjecture can be formulated as the equality

$$\limsup\{L\} = 1.$$

The first purpose of the present note is to comment on a rather evident generalization of the abc-conjecture to a statement involving $n \geq 3$ integers. We show that 1 in the above equality should be replaced by at least $2n-5$. This

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number is also our conjectural value in the "n-conjecture". The second objective of the paper is to present some numerical results concerning deviations of the quotient (1) from the conjectural value 1 in the case of abc-conjecture. Our results do not contradict the conjecture, but the presence of rather big prime factors in the triples a, b, c leading to quotients L strongly deviating from 1 makes it somewhat questionable.

2. The n-conjecture for Z

Let $a_1, a_2, \dots, a_n \in Z$, where $n \geq 3$, satisfy

(i) $\gcd(a_1, a_2, \dots, a_n) = 1$,

(ii) $a_1 + a_2 + \dots + a_n = 0$,

(iii) no proper subsum of (ii) is equal to 0.

Denote

$$M_n = M = \max_{1 \leq j \leq n} (|a_j|), \qquad m_n = m = r(a_1 \cdots a_n), \tag{2}$$
$$L_n = L(a_1, \dots, a_n) = \log M_n / \log m_n.$$

The n-conjecture asserts that, for given $n \geq 3$,

1. the numbers L_n are bounded,

and more precisely

2. $\limsup\{L_n\} = 2n - 5$,

where L_n runs over numbers (2) corresponding to all n-tuples of integers satisfying (i)–(iii).

Theorem 1. *For every* $n \geq 3$,

$$\limsup\{L_n\} \geq 2n - 5.$$

First we prove a lemma.

Lemma 1. *For every* $k \geq 0$, *there exists a polynomial* $f_k \in Z[x]$ *of degree* k *with positive coefficients such that*

$$\frac{x^{2k+1} - 1}{x - 1} = x^k f_k\left(\frac{(x-1)^2}{x}\right). \tag{3}$$

Proof. For $\alpha_j = 2\pi j/(2k+1)$, $j = 1, 2, \dots, k$, we have

$$\frac{x^{2k+1} - 1}{x - 1} = \prod_{j=1}^{k} (x^2 - 2x\cos\alpha_j + 1)$$
$$= x^k \prod_{j=1}^{k} \left(\frac{(x-1)^2}{x} + 2(1 - \cos\alpha_j)\right).$$

It is sufficient to take

$$f_k(z) = \prod_{j=1}^{k} (z + 2(1 - \cos\alpha_j)).$$

From (3) it follows that f_k has integral coefficients, and since all its roots are negative, all its coefficients are positive. □

Remark 1. One can also define the polynomial $f_k(z)$ explicitly:

$$f_k(z) = \sum_{j=0}^{k} \frac{2k+1}{k+j+1} \binom{k+j+1}{2j+1} z^j, \tag{4}$$

or inductively:

$$f_0(z) = 1, \qquad f_1(z) = z + 3,$$

and, for $k \geq 1$,

$$f_{k+1}(z) = (z+2) f_k(z) - f_{k-1}(z). \tag{5}$$

Using (4) or (5), one can continue the list:

$$\begin{aligned}
f_2(z) &= z^2 + 5z + 5, \\
f_3(z) &= z^3 + 7z^2 + 14z + 7, \\
f_4(z) &= z^4 + 9z^3 + 27z^2 + 30z + 9, \\
f_5(z) &= z^5 + 11z^4 + 44z^3 + 77z^2 + 55z + 11, \\
f_6(z) &= z^6 + 13z^5 + 65z^4 + 156z^3 + 182z^2 + 91z + 13.
\end{aligned}$$

As in Lemma 1, one can prove the existence of polynomials $g_k \in Z[x]$ of degree k with positive coefficients such that

$$\frac{x^{2k+2} - 1}{x^2 - 1} = x^k g_k\left(\frac{(x-1)^2}{x}\right)$$

for $k \geq 0$. These polynomials can be defined by a formula similar to (4):

$$g_k(z) = \sum_{j=0}^{k} \binom{k+j+1}{2j+1} z^j, \tag{4'}$$

or inductively by

$$g_0(z) = 1, \qquad g_1(z) = z + 2,$$

and, for $k \geq 1$,

$$g_{k+1}(z) = (z+2) g_k(z) - g_{k-1}(z). \tag{5'}$$

Let us note that the same arguments as in the proof of Lemma 1 give, for $n > 2$,

$$\Phi_n(x) = x^{\phi(n)/2} p_n\left(\frac{(x-1)^2}{x}\right),$$

where Φ_n is the nth cyclotomic polynomial, and $p_n \in Z[x]$ has positive coefficients and degree $\phi(n)/2$ ($\phi(n)$ is the Euler totient function). The splitting field of p_n is the maximal real subfield of the splitting field of Φ_n over the rational numbers. Defining $p_1(x) = p_2(x) = 1$, one can easily prove that f_k and g_k are the products of all polynomials p_d for d dividing $2k+1$, respectively, $2k+2$.

Proof of Theorem 1. Let

$$f_k(z) = \sum_{j=0}^{k} s_j z^j, \tag{6}$$

where according to Lemma 1, the s_j are positive integers. If in (3) we put $k = n-3$ and $x = -a_1/a_2$, then, in view of (6), we get

$$a_1^{2n-5} + a_2^{2n-5} - \sum_{j=0}^{n-3} s_j(a_1+a_2)^{2j+1}(-a_1a_2)^{n-j-3} = 0. \tag{7}$$

If we choose $a_1 = 2^i$, where $i > 1$, and $a_2 = -1$, then we have a sum of n summands equal to zero, with no proper subsum equal to zero, since only the first summand is positive. The second summand is -1, hence the gcd of all summands is 1. Therefore the conditions (i)–(iii) of the n-conjecture are satisfied. With this choice of a_1 and a_2, we have from (7),

$$M_n = 2^{i(2n-5)}.$$

Consequently, denoting $c = 2s_0s_1\cdots s_{n-3}$ and taking the logarithms to the base 2, we get

$$L_n = \frac{i(2n-5)}{\log r((2^i-1)c)} \geq \frac{i(2n-5)}{i+\log r(c)} \longrightarrow 2n-5$$

for $i \to \infty$. Since there are infinitely many i such that the numbers $2^i - 1$ are relatively prime (e.g., all prime i), it is easy to check that the quotients L_n corresponding to those i are different. Therefore, the set $\{L_n\}$ has an accumulation point equal at least $2n-5$. □

Remark 2. Let a_1, a_2, a_3 satisfy the assumptions (i)–(iii) for the 3-conjecture with $a_1 = \max(|a_1|, |a_2|, |a_3|)$ and $L_3 = L(a_1, a_2, a_3)$. If for some $n > 3$, every prime divisor of the coefficients of f_{n-3} divides $a_1a_2a_3$, then (7) gives an example for the n-conjecture with

$$L_n = (2n-5)L_3,$$

since $M_n = a_1^{2n-5}$ and all other terms in (7) are negative.

Thus, the example of E. Reyssat for the 3-conjecture

$$23^5 - 109 \cdot 3^{10} - 2 = 0$$

with $L_3 = 1.629912$ gives the example

$$23^{15} - 109^3 \cdot 3^{30} - 2^3 - 2 \cdot 3^{11} \cdot 23^5 \cdot 109 = 0$$

for the 4-conjecture with $L_4 = 3L_3 = 4.889735$.

The n-conjecture for $K[t]$

Let K be a field of characteristic zero. For a nonzero polynomial $a \in K[t]$, let $r(a)$ be the sum of the degrees of all distinct irreducible factors of a in $K[t]$. Let $a_1, a_2, \ldots, a_n \in K[t]$, where $n \geq 3$, satisfy $\max_{1\leq j\leq n} \deg(a_j) > 0$ and (i)–(iii) as above. Denote

$$\begin{gathered} M_n = M = \max_{1\leq j\leq n} \deg(a_j), \qquad m_n = m = r(a_1\cdots a_n), \\ L_n = L(a_1, \ldots, a_n) = M_n/m_n. \end{gathered} \tag{2'}$$

The n-conjecture asserts that for every $n \geq 3$,

$$M_n \leq (2n-5)(m_n - 1).$$

Theorem 2. *For every* $n \geq 3$,

$$\limsup\{L_n\} \geq 2n - 5.$$

Proof. Put in (7) $a_1 = t^r + 1$, where $r > 0$ and $a_2 = -1$. Then

$$(t^r + 1)^{2n-5} - 1 - t^r \sum_{j=0}^{n-3} s_j t^{2rj}(t^r + 1)^{n-j-3} = 0. \tag{8}$$

Thus, we have a sum of n summands satisfying the assumptions of the n-conjecture. Moreover, for (8), we have

$$M_n = (2n-5)r, \qquad m_n = 1 + r.$$

Consequently,

$$L_n = \frac{(2n-5)r}{1+r} \longrightarrow 2n - 5$$

for $r \to \infty$. □

Remark 3. In the case of polynomial rings an estimation from above is known:

$$L_n \leq \binom{n-1}{2}$$

(see [1], [7] and [8]). Thus, from Theorem 2, we get

Corollary. *If* $n = 3$ *or* 4, *then for the ring* $K[t]$ *we have*

$$\limsup\{L_n\} = 2n - 5.$$

With a suitable modification of the definition of L_n, Theorem 2 and its corollary can be extended to algebraic curves of arbitrary genus over fields of characteristic zero (see [1], [7] and [8]).

4. Examples related to the abc-conjecture

The example of E. Reyssat given above can be interpreted as follows. The equality

$$23^5 - 109 \cdot 9^5 = 2, \qquad \text{i.e.,} \qquad \left(\frac{23}{9}\right)^5 - 109 = \frac{2}{9^5},$$

implies that 23/9 is a good rational approximation to $\sqrt[5]{109}$. Let us consider the continued fraction

$$\sqrt[5]{109} = [2, 1, 1, 4, 77733, \ldots].$$

The very large term 77733 implies that the convergent [2, 1, 1, 4] gives a very good approximation to $\sqrt[5]{109}$. In fact, we have $[2, 1, 1, 4] = 23/9$.

Starting from this observation, we have made an extended computer search for continued fraction expansions of numbers $\sqrt[n]{k}$. Having a suitable convergent of the continued fraction of $\sqrt[n]{k}$, say, p/q, we put $c = \max(kq^n, p^n)$, $b = \min(kq^n, p^n)$, $a = c - b$ (divided by $\gcd(a, b, c)$). We have also considered several rational numbers p/q which can be derived from the convergents of continued fractions, and which give good approximations to $\sqrt[n]{k}$ such that p and q have many prime power divisors.

The "obvious" idea was that if $[a_0, a_1, \ldots]$ is the fraction, then one should look for the convergents corresponding to large a_i in order to get a good approximation. Then we looked for large q in the convergents p/q (which is more reasonable). But it appears that these properties are not relevant in general. For example, the best-known result $L = 1.629912$ can be obtained not only from $\sqrt[5]{109}$ but also from $\sqrt{2507} = [50, 14, 3, 2, 1, 1, 1, 1, \dots]$ and the convergent of length 6 equal to $23^3/3^5$.

Using this method, we obtained several new interesting examples (indicated B-B) and all previously known. All results with $L > 1.4$ known to us at present (March 15, 1993) are included in the table. It contains the examples given by B.M.M. de Weger in [6] and the examples constructed by A. Nitaj in [3]. We express our thanks to A. Nitaj for sending us his examples, which were obtained by a different method. We have also included one example of Xiao Gang (sent to us by B.M.M. de Weger, see also Oesterlé [4]) and one of J. Kanapka (sent to us by N. Elkies).

TABLE
(version of March 15, 1993)

1.	1.629912	$2 + 3^{10} \cdot 109$	=	23^5	E. Reyssat
2.	1.625991	$11^2 + 3^2 \cdot 5^6 \cdot 7^3$	=	$2^{21} \cdot 23$	B. M. M. de Weger
3.	1.623490	$19 \cdot 1307 + 7 \cdot 29^2 \cdot 31^8$	=	$2^8 \cdot 3^{22} \cdot 5^4$	B-B
4.	1.580756	$283 + 5^{11} \cdot 13^2$	=	$2^8 \cdot 3^8 \cdot 17^3$	B-B, A. Nitaj
5.	1.567887	$1 + 2 \cdot 3^7$	=	$5^4 \cdot 7$	B. M. M. de Weger
6.	1.547075	$7^3 + 3^{10}$	=	$2^{11} \cdot 29$	B. M. M. de Weger
7.	1.526999	$13 \cdot 19^6 + 2^{30} \cdot 5$	=	$3^{13} \cdot 11^2 \cdot 31$	A. Nitaj
8.	1.502839	$239 + 5^8 \cdot 17^3$	=	$2^{10} \cdot 37^4$	B-B, A. Nitaj
9.	1.497621	$5^2 \cdot 7937 + 7^{13}$	=	$2^{18} \cdot 3^7 \cdot 13^2$	B. M. M. de Weger
10.	1.492432	$2^2 \cdot 11 + 3^2 \cdot 13^{10} \cdot 17 \cdot 151 \cdot 4423$	=	$5^9 \cdot 139^6$	A. Nitaj
11.	1.491590	$73 + 2^{13} \cdot 7^7 \cdot 941^2$	=	$3^{16} \cdot 103^3 \cdot 127$	A. Nitaj
12.	1.488865	$11^2 + 3^9 \cdot 13$	=	$2^{11} \cdot 5^3$	B. M. M. de Weger
13.	1.482910	$37 + 2^{15}$	=	$3^8 \cdot 5$	B. M. M. de Weger
14.	1.474450	$1 + 3^{16} \cdot 7$	=	$2^3 \cdot 11 \cdot 23 \cdot 53^3$	B-B, A. Nitaj
15.	1.474137	$7^2 + 2^{10} \cdot 11 \cdot 53^2$	=	$3^4 \cdot 5^8$	B-B, A. Nitaj
16.	1.471298	$3^4 \cdot 199 + 11^8$	=	$2^3 \cdot 5^7 \cdot 7^3$	B-B, A. Nitaj
17.	1.461924	$2^7 \cdot 5^2 + 7^6 \cdot 41$	=	13^6	B. M. M. de Weger
18.	1.457066	$3^2 \cdot 5^2 + 2^4 \cdot 17^3 \cdot 31^4$	=	$7^{10} \cdot 257$	B-B, A. Nitaj
19.	1.455673	$1 + 2^5 \cdot 3 \cdot 5^2$	=	7^4	B. M. M. de Weger
20.	1.455126	$3^2 \cdot 11^6 + 2^{35}$	=	$19^5 \cdot 13883$	B-B
21.	1.452613	$2^{19} \cdot 13 \cdot 103 + 7^{11}$	=	$3^{11} \cdot 5^3 \cdot 11^2$	B. M. M. de Weger
22.	1.451344	$3^5 \cdot 7 + 5^6 \cdot 67$	=	2^{20}	B-B, A. Nitaj
23.	1.450858	$3^5 \cdot 7^3 + 2^{13} \cdot 23^3 \cdot 59$	=	$5^3 \cdot 19^6$	B-B
24.	1.450026	$1 + 3^3 \cdot 5^3 \cdot 7^7 \cdot 23$	=	$2^{13} \cdot 11^4 \cdot 13 \cdot 41$	A. Nitaj
25.	1.449651	$1 + 3 \cdot 5^5 \cdot 47^2$	=	$2^{18} \cdot 79$	G. Frey
26.	1.447977	$11^2 \cdot 43 + 5^9 \cdot 7^2 \cdot 13^4 \cdot 97$	=	$2^3 \cdot 3 \cdot 73^7$	A. Nitaj
27.	1.447743	$89 + 7 \cdot 11^8$	=	$2^{20} \cdot 3^3 \cdot 53$	B-B, A. Nitaj
28.	1.446246	$3^2 \cdot 5^7 \cdot 79 + 2^{29} \cdot 13$	=	$11^7 \cdot 19^2$	A. Nitaj
29.	1.445064	$2 \cdot 13^2 + 5^8$	=	$3 \cdot 19^4$	B-B, A. Nitaj
30.	1.443307	$1 + 2^{12} \cdot 5^3$	=	$3^5 \cdot 7^2 \cdot 43$	B. M. M. de Weger
31.	1.443284	$3^2 \cdot 19^3 + 5^{11}$	=	$2^{17} \cdot 373$	B-B, A. Nitaj
32.	1.441441	$31^3 + 2 \cdot 17 \cdot 41^5$	=	$3 \cdot 5^7 \cdot 7^5$	B-B, A. Nitaj
33.	1.440969	$3^4 \cdot 23^2 + 31^5$	=	$2^{15} \cdot 5^3 \cdot 7$	B-B, A. Nitaj
34.	1.439063	$1 + 2^4 \cdot 3^7 \cdot 547$	=	$5^8 \cdot 7^2$	B. M. M. de Weger
35.	1.438360	$1 + 19 \cdot 509^3$	=	$2^{19} \cdot 3^4 \cdot 59$	B-B
36.	1.436180	$2 \cdot 13^5 + 7^6 \cdot 173^2$	=	$3^{13} \cdot 47^2$	A. Nitaj
37.	1.435006	$2^{10} \cdot 7 + 5^7$	=	$3^8 \cdot 13$	B. M. M. de Weger
38.	1.433464	$2^5 \cdot 3^{18} + 5^6 \cdot 7^{10} \cdot 23^2$	=	$11^9 \cdot 691 \cdot 1433$	A. Nitaj
39.	1.433043	$31^2 + 3^5 \cdot 5^9$	=	$2^5 \cdot 23^4 \cdot 53$	B-B, A. Nitaj
40.	1.432904	$2^{21} + 7^6 \cdot 17 \cdot 8209^2$	=	$5^{12} \cdot 743^2$	A. Nitaj
41.	1.431092	$2^9 \cdot 19^2 + 3^3 \cdot 5^7 \cdot 7^2 \cdot 31^3$	=	$59^6 \cdot 73$	A. Nitaj
42.	1.430418	$193 + 2 \cdot 5^6 \cdot 19^2 \cdot 1193^2$	=	$3^9 \cdot 13^8$	B-B, A. Nitaj
43.	1.430176	$3^6 \cdot 7^2 \cdot 13 \cdot 127^2 + 2^{38} \cdot 61 \cdot 137$	=	$5^{11} \cdot 19^6$	B-B
44.	1.429552	$3^9 \cdot 29 + 7^6 \cdot 43^2$	=	$2^{24} \cdot 13$	A. Nitaj
45.	1.429007	$3^{21} + 7^2 \cdot 11^6 \cdot 199$	=	$2 \cdot 13^8 \cdot 17$	A. Nitaj

46.	1.428908	$73^2 + 2^{11} \cdot 11^4 \cdot 13^3 = 3^{11} \cdot 5^5 \cdot 7 \cdot 17$	B-B
47.	1.428323	$11 + 7^3 \cdot 167^2 = 2 \cdot 3^{14}$	B-B, A. Nitaj
48.	1.427566	$73 + 11^5 \cdot 157^2 = 2^2 \cdot 3^{10} \cdot 7^5$	B-B, A. Nitaj
49.	1.427488	$61^4 + 2^{20} \cdot 41^3 \cdot 83^2 = 3^{22} \cdot 5 \cdot 19 \cdot 167$	A. Nitaj
50.	1.427115	$3^{10} + 7^8 \cdot 23 = 2^9 \cdot 509^2$	A. Nitaj
51.	1.426753	$31 + 2^5 \cdot 5^{10} \cdot 19^2 = 3 \cdot 7^5 \cdot 11^3 \cdot 41^2$	B-B, A. Nitaj
52.	1.426565	$3 + 5^3 = 2^7$	B. M. M. de Weger
53.	1.423381	$5^2 \cdot 11 + 13^3 \cdot 1483^2 = 2^{29} \cdot 3^2$	B-B, A. Nitaj
54.	1.421828	$2^4 \cdot 59 + 5^{12} \cdot 19 = 3^3 \cdot 11^2 \cdot 17^5$	B-B, A. Nitaj
55.	1.421575	$5^7 + 11^5 \cdot 13^2 = 2^{15} \cdot 7^2 \cdot 17$	B-B, A. Nitaj
56.	1.421008	$2^9 \cdot 37^3 \cdot 89 + 3^9 \cdot 5^9 \cdot 31 = 103^6$	B-B, A. Nitaj
57.	1.420437	$7^8 \cdot 19 + 2^{15} \cdot 5^2 \cdot 37^2 = 3 \cdot 17^7$	A. Nitaj
58.	1.420036	$23^3 + 3^9 \cdot 5^7 \cdot 31 = 2^7 \cdot 7^3 \cdot 13 \cdot 17^4$	A. Nitaj
59.	1.418919	$7^2 + 2^{17} \cdot 181^2 = 3^8 \cdot 809^2$	B-B, A. Nitaj
60.	1.418233	$13 \cdot 3499 + 2^{39} = 3^4 \cdot 5^{11} \cdot 139$	B-B
61.	1.417633	$5^6 \cdot 1609 + 2^9 \cdot 3^{14} \cdot 13^3 = 1523^4$	B-B
62.	1.416793	$3^9 \cdot 43^3 + 5^{13} \cdot 5323 = 2^7 \cdot 7^3 \cdot 23^6$	A. Nitaj
63.	1.416438	$41^4 \cdot 33941 + 3^{12} \cdot 19^7 = 2^{23} \cdot 5^9 \cdot 29$	B-B
64.	1.416051	$3 \cdot 5^4 \cdot 599 + 11 \cdot 23^8 = 2^{22} \cdot 59^3$	B-B, A. Nitaj
65.	1.415561	$7^3 + 5^{13} \cdot 181 = 2^4 \cdot 3 \cdot 11 \cdot 13^2 \cdot 19^5$	A. Nitaj
66.	1.414503	$3^{11} \cdot 5^4 + 7 \cdot 11^6 \cdot 43 = 2^{17} \cdot 17^3$	Xiao Gang
67.	1.413698	$2^6 \cdot 5 \cdot 137 + 3^{14} = 13^6$	B-B, A. Nitaj
68.	1.413279	$5^2 + 3^7 \cdot 13^3 = 2^8 \cdot 137^2$	B-B, A. Nitaj
69.	1.413166	$3^6 \cdot 157^3 \cdot 283 + 23^{10} = 2^{30} \cdot 5^2 \cdot 11^2 \cdot 13$	B-B, A. Nitaj
70.	1.412681	$5 + 3^{11} = 2^{10} \cdot 173$	B. M. M. de Weger
71.	1.411680	$79^3 + 3^6 \cdot 7 \cdot 11 \cdot 13^5 = 2^{18} \cdot 43^3$	A. Nitaj
72.	1.411615	$3 \cdot 13^2 \cdot 1049 + 2^{39} \cdot 29^2 \cdot 107 = 19^3 \cdot 139^6$	B-B, A. Nitaj
73.	1.410683	$67^2 \cdot 2399 + 3^{13} \cdot 107^3 = 2^6 \cdot 5^{15}$	B-B
74.	1.410044	$2^{13} \cdot 3^{13} \cdot 11^3 + 13 \cdot 29 \cdot 43^6 \cdot 673 = 5^{20} \cdot 17$	A. Nitaj
75.	1.408973	$7^2 + 83^5 = 2^2 \cdot 3^{12} \cdot 17 \cdot 109$	B-B, A. Nitaj
76.	1.407787	$2^2 \cdot 13 + 7^3 \cdot 41^5 \cdot 181 = 3^{14} \cdot 5 \cdot 67^3$	A. Nitaj
77.	1.407404	$3^2 \cdot 233 + 23^7 \cdot 293^2 = 2^{15} \cdot 5^2 \cdot 13^5 \cdot 31^2$	A. Nitaj
78.	1.407208	$241 + 2^{12} \cdot 3^4 \cdot 5^6 \cdot 1181 = 11^8 \cdot 13^4$	B-B
79.	1.407051	$3^9 \cdot 163 + 2^3 \cdot 11^6 \cdot 17 = 5^{12}$	B-B, A. Nitaj
80.	1.406524	$7^9 + 3^2 \cdot 5^7 \cdot 13^3 = 2^{16} \cdot 19^2 \cdot 67$	J. Kanapka
81.	1.406420	$2^{19} \cdot 367^3 + 5^{17} \cdot 197 \cdot 281 = 13^2 \cdot 251^6$	A. Nitaj
82.	1.406097	$2^{16} \cdot 41 \cdot 71 + 3^{15} \cdot 7^2 = 19^7$	A. Nitaj
83.	1.406079	$5 \cdot 7^2 + 13^2 \cdot 43^3 = 2^{11} \cdot 3^8$	B-B, A. Nitaj
84.	1.405785	$13^3 + 2^9 \cdot 37^2 = 3^2 \cdot 5^7$	B-B, A. Nitaj
85.	1.405443	$2^{24} \cdot 3^5 + 5 \cdot 19^5 \cdot 59^2 = 7^{10} \cdot 167$	A. Nitaj
86.	1.404484	$631 + 2^{26} \cdot 5 \cdot 29^2 = 3^3 \cdot 7^{10} \cdot 37$	B-B, A. Nitaj
87.	1.404264	$1 + 3^9 \cdot 7^2 \cdot 197 = 2^7 \cdot 5^7 \cdot 19$	B-B, A. Nitaj
88.	1.403482	$3^3 \cdot 13 + 2^5 \cdot 11 \cdot 19^2 \cdot 73^3 = 5^2 \cdot 7^{11}$	A. Nitaj
89.	1.402183	$3^{12} \cdot 5^6 + 7^9 \cdot 31^2 = 2^9 \cdot 11^5 \cdot 571$	A. Nitaj
90.	1.401979	$2^{33} \cdot 5 + 3^9 \cdot 7^6 \cdot 31^2 \cdot 97 = 11^2 \cdot 19^3 \cdot 127^4$	A. Nitaj

Some words about the program. The examples are constructed with $\sqrt[n]{k}$, where $2 \le k \le 2 \cdot 10^5$, $2 \le n \le 15$ (for $k \le 100$, we choose n up to 20, but the increase of n has not resulted in new examples). The computations were carried out with all convergents up to length 10 (for $k \le 100$ up to 20 without new examples). In order to limit the computation time, we put the restriction $c < 10^{15}$ (in some intervals for k, we took $c < 10^{30}$).

Of course, there is nothing which makes it impossible to continue computations of new examples by using the same method. But it is much more desirable to understand why so many examples with large values of L can be constructed in such a way. The first of the three remarks concluding the paper is closely related to this question.

Remark 4. As we noted before, all examples in the table can be obtained by using continued fractions of $\sqrt[n]{k}$ for suitable n and k. In order to check this possibility, let us introduce the following notations. If x is a positive integer, let $n(x)$ be the largest exponent of prime numbers dividing x, and for $s(x) \geq n(x)$, let $x'_{s(x)}$ be the unique integer such that $xx'_{s(x)} = r(x)^{s(x)}$. We shall write x' when $s(x)$ is clear from the context. With these notations, we have the following easy result:

Lemma 2. *Let a, b, c be positive integers such that $a + b = c$, and $a = \rho b$, where $0 < \rho < 1$. If*

$$\frac{1}{\rho} < \frac{s(a)}{r(a)} \quad \text{or} \quad \rho < \frac{s(b)}{r(b)} \quad \text{or} \quad \rho < \frac{s(c)}{2r(c)},$$

then $r(a)$ is a convergent of $\sqrt[s(a)]{a'c}$, or $r(b)$ is a convergent of $\sqrt[s(b)]{b'c}$, or $r(c)$ is a convergent of $\sqrt[s(c)]{bc'}$, respectively.

Proof. Consider the third case, that is, $\rho < \frac{s(c)}{2r(c)}$. Using the mean value theorem, we get

$$r(c) - \sqrt[s(c)]{bc'} \leq \frac{\sqrt[s(c)]{bc'}}{s(c)bc'}(cc' - bc') \leq \frac{r(c)a}{s(c)b} < \frac{1}{2}.$$

Thus, $r(c)$ is a convergent of $\sqrt[s(c)]{bc'}$ (in fact, the second one). Similar arguments show that in the first case (or in the second, with a replaced by b), $\sqrt[s(a)]{a'c} - r(a) < 1$, so $r(a)$ is the first convergent of $\sqrt[s(a)]{a'c}$. □

Using Lemma 2, we can easily check that its assumptions are satisfied for almost all the examples in the table with $s(x) = n(x)$ for $x \in \{a, b, c\}$ (in fact for all but five examples with $x = b$ or c). In any case, one can choose a sufficiently large value of, say, $s(c)$, in order to fulfill these assumptions. Then, according to our algorithm, we get all the examples using the roots and their convergents given by the lemma. Of course, such a choice of n and k in $\sqrt[n]{k}$ is not always the optimal one.

Remark 5. There are other quotients, similar to (2), which are natural in connection with the *abc*-conjecture. Following [4] and [5], we let

$$L' = L'(a, b, c) = \frac{\log|abc|}{\log r(abc)},$$

for relatively prime nonzero integers a, b, c such that $a + b = c$. It is evident that the *abc*-conjecture implies the inequality

$$\limsup\{L'\} \leq 3.$$

The deviations of the quotients L' from 3 have been studied intensively by A. Nitaj (see [3]). The biggest value $L' = 4.419014$ corresponds to Nitaj's Example 7 in the table. It is a better result than $L' = 4.107567$ corresponding to the example of Xiao Gang cited in [4] (Example 66 in the table).

Remark 6. We observe that in all the examples in the table, the exponent of at least one of the prime numbers involved is ≤ 2. If x is a nonzero integer, we say that x is n-powerful if p^n divides x for each prime number p dividing x (2-powerful numbers are usually called powerful—see, e.g., [2, B16]). With this terminology, we do not have an example of 3-powerful integers a, b, c such that $a + b = c$, $\gcd(a, b, c) = 1$ and $L > 1.4$ (or even with $L > 1.2$). However,

$$271^3 + 2^3 \cdot 3^5 \cdot 73^3 = 919^3$$

with not impressive L. We do not know whether there are 4-powerful a, b, c such that $a+b=c$ and $\gcd(a, b, c)=1$. But there are reasons to believe that there are no n-powerful integers satisfying these conditions when $n \geq 5$. In fact, our computations strongly suggest that

$$\max(|a|, |b|, |c|) \leq r(abc)^s$$

with $s < 1.65$. If this is true, then for n-powerful numbers a, b, c, we get $r(abc) \leq \sqrt[n]{|abc|}$. Therefore, $|abc|^n \leq |abc|^{3s} < |abc|^5$, so $n < 5$.

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