Integer factorization,
part 1: the $\mathbf{Q}$ sieve

Integer factorization,
part 2: detecting smoothness

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The $\mathbf{Q}$ sieve factors n
by combining enough
y-smooth congruences $i(n+i)$.

"Enough" $\approx$ "$> y/\log y$."
Plausible conjecture: if $y \in$
$\exp\sqrt{\left(\frac{1}{2}+o(1)\right)\log n \log\log n}$
then $y^{2+o(1)}$ congruences
have enough smooth congruences.

Linear sieve, quadratic sieve,
number-field sieve: similar.

How to figure out
which congruences are smooth?

Could use trial division:
For each congruence,
remove factors of 2,
remove factors of 3,
remove factors of 5,
etc.; use all primes $p \le y$.

$y^{3+o(1)}$ bit operations:
$y^{1+o(1)}$ for each congruence.
Want something faster!

Textbook answer: Sieving.

Generate in order of p,
then sort in order of i,
all pairs (i, p) with
i in range and $i(n + i) \in p\mathbf{Z}$.

Pairs for one p are
(p, p), $(2p, p)$, $(3p, p)$, etc.
and $(p - (n \bmod p), p)$ etc.

$(\lg y)^{O(1)}$ bit operations
for each congruence.

Do record-setting factorizations use the textbook answer? No!

Sieving has two big problems.

First problem:
Sieving needs large i range, $\geq y^{1+o(1)}$ consecutive values. Limits number of sublattices, so limits smoothness chance.

Can eliminate this problem using "remainder trees."

Given $c_1, c_2, \ldots, c_m$,
together having $y(\lg y)^{O(1)}$ bits:

Can compute $c_1 c_2 \cdots c_m$
with $y(\lg y)^{O(1)}$ operations.

Actually compute
"product tree" of $c_1, c_2, \ldots, c_m$.
Root: $c_1 c_2 \cdots c_m$.
Left subtree if $m \geq 2$:
product tree of $c_1, \ldots, c_{\lceil m/2 \rceil}$.
Right subtree if $m \geq 2$:
product tree of $c_{\lceil m/2 \rceil + 1}, \ldots, c_m$.

e.g. tree for 23, 29, 84, 15, 58, 19:

$$\begin{tikzcd}
 & & & & & 926142840 & & & & & \\
 & & & 56028 \arrow[urr] & & & & & 16530 \arrow[ulll] & & \\
 & 667 \arrow[urr] & & & 84 \arrow[ul] & & & 870 \arrow[ur] & & & 19 \arrow[ull] \\
23 \arrow[ur] & & 29 \arrow[ul] & & & & 15 \arrow[ur] & & 58 \arrow[ul] & &
\end{tikzcd}$$

Obtain each level of tree
with $y(\lg y)^{O(1)}$ operations
by multiplying lower-level pairs.
Use FFT-based multiplication.

Remainder tree

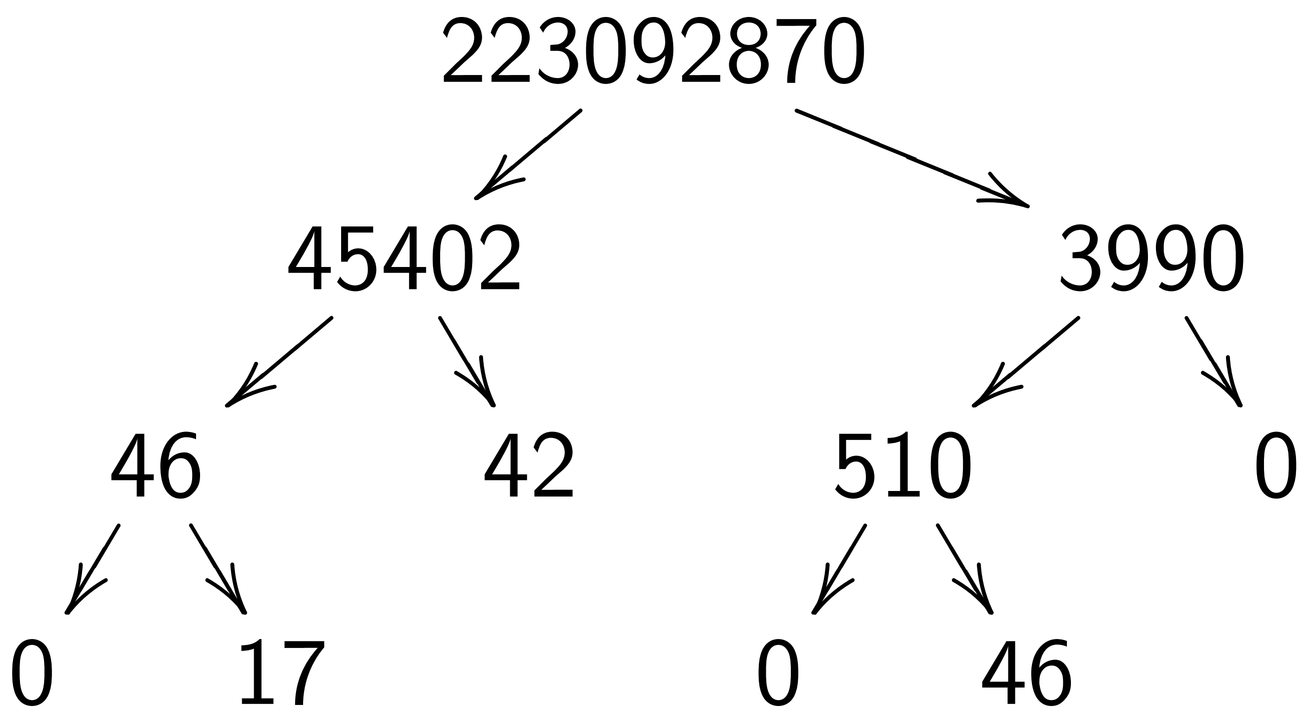
of $P, c_1, c_2, \ldots, c_m$ has one

node $P \bmod C$ for each node C

in product tree of $c_1, c_2, \ldots, c_m$.

e.g. remainder tree of

$223092870, 23, 29, 84, 15, 58, 19$:



Use product tree to compute product P of primes $p \le y$.

Use remainder tree to compute $P \bmod c_1, P \bmod c_2, \ldots$.

Now c_1 is y-smooth
iff $P^{2^k} \bmod c_1 = 0$ for
minimal $k \ge 0$ with $2^{2^k} \ge c_1$.
Similarly c_2 etc.

Total $y(\lg y)^{O(1)}$ operations
if $c_1, c_2, \ldots$ together
have $y(\lg y)^{O(1)}$ bits.

Second problem with sieving,
not fixed by remainder trees:
Need $y^{1+o(1)}$ bits of storage.

Real machines don't have much
fast memory: it's expensive.

Effect is not visible for
small computations on
single serial CPUs,
but becomes critical in
huge parallel computations.

How to quickly find primes
above size of fast memory?

The rho method

Define $\rho_0 = 0$, $\rho_{k+1} = \rho_k^2 + 11$.

Every prime $\le 2^{20}$ divides $S = (\rho_1 - \rho_2)(\rho_2 - \rho_4)(\rho_3 - \rho_6) \cdots (\rho_{3575} - \rho_{7150})$.
Also many larger primes.

Can compute $\gcd\{c, S\}$ using $\approx 2^{14}$ multiplications mod c, very little memory.

Compare to $\approx 2^{16}$ divisions for trial division up to 2^{20}.

More generally: Choose z.
Compute $\gcd\{c, S\}$ where $S = (\rho_1 - \rho_2)(\rho_2 - \rho_4) \cdots (\rho_z - \rho_{2z})$.

How big does z have to be for all primes $\leq y$ to divide S?

Plausible conjecture: $y^{1/2+o(1)}$; so $y^{1/2+o(1)}$ mults mod c.

Reason: Consider first collision in $\rho_1 \bmod p, \rho_2 \bmod p, \ldots$.
If $\rho_i \bmod p = \rho_j \bmod p$ then $\rho_k \bmod p = \rho_{2k} \bmod p$ for $k \in (j - i)\mathbf{Z} \cap [i, \infty] \cap [j, \infty]$.

The $p-1$ method

Have built an integer S
divisible by all primes $\leq y$.
Less costly way to do this?

First attempt:
Define $S_1 = 2^{E(z)} - 1$ where
$E(z) = \text{lcm}\{1, 2, 3, \ldots, z\}$.

If $E(z) \in (p-1)\mathbf{Z}$ then $S_1 \in p\mathbf{Z}$.

Can tweak to find more p's:
e.g., could instead use product
of $2^{E(z)} - 1$ and $2^{E(z)q} - 1$
for all primes $q \in [z+1, z \log z]$;
could replace $E(z)$ by $E(z)^2$.

e.g. $z = 20$:

$$E(z) = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$$
$$= 232792560.$$

$S_1 = 2^{E(z)} - 1$ has prime divisors 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 53, 61, 67, 71, 73, 79, 89, 97, 103, 109, 113, 127, 131, 137, 151, 157, 181, 191, 199, etc.

Compute S_1 with 34 mults.

As $z \to \infty$: $(1.44\ldots + o(1))z$
multiplications to compute S_1.

Dividing $E(z)$ is stronger
than z-smoothness but not much.

Plausible conjecture: if $z \in$
$\exp\sqrt{\left(\frac{1}{2}+o(1)\right)\log y \log\log y}$
then $p-1$ divides $E(z)$
with chance $1/z^{1+o(1)}$
for uniform random prime $p \le y$.

So method finds some primes
at surprisingly high speed.
What about the other primes?

The $p+1$ method

Second attempt:

Define $v_0 = 2$, $v_1 = 10$,

$v_{2i} = v_i^2 - 2$,

$v_{2i+1} = v_i v_{i+1} - v_1$.

Define $S_2 = v_{E(z)} - 2$.

Point of v_i formulas:

$v_i = \alpha^i + \alpha^{-i}$

in $\mathbf{Z}[\alpha]/(\alpha^2 - 10\alpha + 1)$.

If $E(z) \in (p+1)\mathbf{Z}$

and $10^2 - 4$ non-square in $\mathbf{F}_p$

then $\mathbf{F}_p[\alpha]/(\alpha^2 - 10\alpha + 1)$

is a field so $S_2 \in p\mathbf{Z}$.

e.g. $z = 20$, $E(z) = 232792560$:

$S_2 = v_{E(z)} - 2$ has prime divisors 3, 5, 7, 11, 13, 17, 19, 23, 29, 37, 41, 43, 53, 59, 67, 71, 73, 79, 83, 89, 97, 103, 109, 113, 131, 151, 179, 181, 191, 211, 227, 233, 239, 241, 251, 271, 307, 313, 331, 337, 373, 409, 419, 439, 457, 467, 547, 569, 571, 587, 593, 647, 659, 673, 677, 683, 727, 857, 859, 881, 911, 937, 967, 971, etc.

The elliptic-curve method

Fix $a \in \{6, 10, 14, 18, \ldots\}$.

Define $x_1 = 2$, $d_1 = 1$,
$x_{2i} = (x_i^2 - d_i^2)^2$,
$d_{2i} = 4x_i d_i (x_i^2 + a x_i d_i + d_i^2)$,
$x_{2i+1} = 4(x_i x_{i+1} - d_i d_{i+1})^2$,
$d_{2i+1} = 8(x_i d_{i+1} - d_i x_{i+1})^2$.

Define $S_a = d_{E(z)}$.

Have now supplemented S_1, S_2
with S_6, S_{10}, S_{14}, etc.
Variability of a is important.

Point of x_i, d_i formulas:

If $d_i(a^2-4)(4a+10) \notin p\mathbf{Z}$
then ith multiple of $(2,1)$
on the elliptic curve
$(4a+10)y^2 = x^3+ax^2+x$
over $\mathbf{F}_p$ is $(x_i/d_i, \ldots)$.

If $(a^2-4)(4a+10) \notin p\mathbf{Z}$
and $E(z) \in (\text{order of } (2,1))\mathbf{Z}$
then $S_a \in p\mathbf{Z}$.

Order of elliptic-curve group
depends on a but is always
in $[p+1-2\sqrt{p}, p+1+2\sqrt{p}]$.

Consider smallest z
such that product of S_a
for first z choices of a
is divisible by every $p \le y$.

Plausible conjecture: $z \in$
$\exp\sqrt{\left(\frac{1}{2}+o(1)\right)\log y \log\log y}$.

Computing this product
takes $\approx 12z^2$ mults; i.e.
$\exp\sqrt{(2+o(1))\log y \log\log y}$.

Early aborts

Neverending supply
of congruences

$\downarrow$ initial selection

Smallest congruences

$\downarrow$

Partial factorizations
using primes $\leq y^{1/2}$

$\downarrow$ early abort

Smallest unfactored parts

$\downarrow$

Partial factorizations
using primes $\leq y$

$\downarrow$ final abort

Smooth congruences

Say we use trial division.
Time $y^{1/2+o(1)}$ for $\le y^{1/2}$.
Time $y^{1+o(1)}$ for $\le y$.

Say we choose "smallest"
so that each congruence
has chance $y^{1/2+o(1)}/y^{1+o(1)}$
of surviving early abort.
Fact: A y-smooth congruence
has chance $y^{-1/4+o(1)}$
of surviving early abort.

Have reduced trial-division
time by factor $y^{1/2+o(1)}$.
Have reduced identify-a-smooth
time by factor $y^{1/4+o(1)}$.

More generally, can abort at
$y^{1/k}$, $y^{2/k}$, etc.
to reduce trial-division
time by factor $y^{1-1/k+o(1)}$.
This reduces identify-a-smooth
time by factor $y^{(1-1/k)/2+o(1)}$.

Generalize beyond trial division
to sieving, remainder trees,
trial division, rho, ECM.

Use many aborts to combine
many methods into one
grand unified method
for smoothness detection.

Are all primes small?

Instead of using these methods to find smooth congruences c, can apply them directly to n.

Worst case: n is product of two primes $\approx \sqrt{n}$.

Take $y \approx \sqrt{n}$.
Number of mults mod n in elliptic-curve method:
$\exp\sqrt{(2+o(1))\log y \log\log y} =$
$\exp\sqrt{(1+o(1))\log n \log\log n}$.

Faster than $\mathbf{Q}$ sieve.
Comparable to quadratic sieve, using much less memory.

Slower than number-field sieve for sufficiently large n.

One elliptic-curve computation found a prime $\approx 2^{219}$ in $\approx 3 \cdot 10^{12}$ Opteron cycles.
Fairly lucky in retrospect.